Lecture #3 September 2, 2004

# Pulsar & Neutron stars

## Magnetic dipole radiation formula

Energy loss rate due to magnetic dipole radiation —

$$\frac{dE}{dt} = -\frac{2}{3}\frac{\ddot{m}^2}{c^3}$$

(where $m$ is magnetic dipole moment the magnetic field in terms of $m$ is given by $\vec{B} = \frac{3\hat{r}(\hat{r}\cdot\vec{m}) - \vec{m}}{r^3}$ [See Jackson 5.56]

$\therefore\ m \sim BR^3/2$ ; $B$ : mag field at NS surface & $R$ is n-star radius.)

$\ddot{m} \approx m\Omega^2 \sin\alpha$ ; $\alpha$ is the angle between $\vec{\Omega}$ & $\vec{m}$.

$$\frac{dE}{dt} = -\frac{2}{3}\frac{m^2\Omega^4\sin^2\alpha}{c^3} = -\frac{1}{6}\frac{B^2R^6\Omega^4\sin^2\alpha}{c^3}$$

For Crab pulsar
$\Omega = \frac{2\pi}{.033\,s} = 190$ rad $s^{-1}$
$B \sim 5\times10^{12}$ G
$\therefore \frac{dE}{dt} \sim 2\times10^{38}$ erg $s^{-1}$

for $\Omega$ 10 x larger & B $10^3$ larger
$\dot{E} \sim 2\times10^{48}$ erg $s^{-1}$

$$E = \frac{1}{2}I\Omega^2$$

or $$\frac{d\Omega}{dt} = -\frac{B^2R^6\Omega^3\sin^2\alpha}{6Ic^3}$$

Observationally it is found that $\frac{d\Omega}{dt} \propto \Omega^a$ with 'a' as small as 1.4 for some pulsar & $\sim 2.8$ for some others.
(This suggests that there is a torque on NS due to some process other than mag. dipole radiation. It could be for instance due to particle outflow.)

Take $$\frac{d\Omega}{dt} = -K\Omega^a \Rightarrow \frac{1}{\Omega^{a-1}(t)} - \frac{1}{\Omega^{a-1}(t_0)} = K(a-1)(t-t_0)$$

for $\Omega(t) \ll \Omega(t_0)$, the pulsar spin-down age —

$$\boxed{(t-t_0) = \frac{1}{K\Omega^{a-1}(t)(a-1)} = \frac{\Omega}{|\dot{\Omega}|(a-1)}}$$

Turns out to be OK for many of the historical SNe pulsars

## Pulsar distance determination (dispersion measure)

$$\omega^2 = \omega_p^2 + k^2c^2$$

$$\omega_p^2 = \frac{4\pi n_e e^2}{m_e} = 3\times10^9\, n_e \quad \text{or} \quad \nu_p = \frac{\omega_p}{2\pi} = 8.5\, n_e^{1/2}\ \text{KHz}$$

time for a pulse to arrive

$$t_{obs}(\omega) = \int_0^d \frac{dl}{V_g} \quad ; \quad V_g = \frac{d\omega}{dk} = \frac{kc^2}{\omega} = \frac{c}{\omega}\sqrt{\omega^2 - \omega_p^2} = c\sqrt{1-\frac{\omega_p^2}{\omega^2}}$$

$$\approx c\left(1 - \frac{\omega_p^2}{2\omega^2}\right)$$

$$t_{obs}(\omega) = \frac{d}{c} + \frac{1}{2c\omega^2}\int dl\, \omega_p^2 = \frac{d}{c} + \frac{2\pi e^2}{c m_e \omega^2}\underbrace{\int_0^d dl\, n_e}_{\equiv \text{DM (dispersion measure)}}$$

or $$t_{obs}(\omega) = \frac{d}{c} + \frac{2\pi e^2}{c m_e \omega^2}\langle n_e\rangle d$$

$$\frac{dt_{obs}}{d\omega} = -\frac{4\pi e^2}{c m_e \omega^3}\langle n_e\rangle d$$

or $$\boxed{d = \frac{c m_e \omega^3}{4\pi e^2}\left|\frac{dt_{obs}}{d\omega}\right|\frac{1}{\langle n_e\rangle}}$$

$\langle n_e\rangle \approx 0.03\ \text{cm}^{-3}$ in the Galaxy; but it varies along different lines of sight & different part of the Galaxy. $\langle n_e\rangle$ is determined from pulsars for which we know the distance in an independent way.

## Pulsar Magnetosphere

- First calculate scale height for NS & show that the particle density is vanishingly small just a few meters from the surface of a NS.

### Goldreich-Julian Model

- The interior of a NS is highly conducting fluid so there should be no E-field in its co-rotating frame (current $\vec{J} = \sigma \vec{E}$; $\sigma$ is so large that a small $\vec{E}$ will cause a huge current).

There must be $\vec{E}$ field in the NS as seen by an inertial frame observer; call this field to be $\vec{E}$. Since the field in the comoving frame is zero —

$$\Rightarrow \quad \vec{E} + \frac{\vec{V}}{c} \times \vec{B} = 0 \qquad \vec{V} = \vec{\Omega} \times \vec{r}$$

$$\text{or} \quad \vec{E} = -\frac{(\vec{\Omega} \times \vec{r}) \times \vec{B}}{c} = \frac{(\vec{B}\cdot\vec{r})\vec{\Omega} - (\vec{B}\cdot\vec{\Omega})\vec{r}}{c}$$

The $\theta$-component of the $\vec{E}$ at the surface of the NS is —

$$E_\theta(R) = -\frac{(\vec{B}\cdot\vec{r})}{c}\Omega \sin\theta$$

for dipole magnetic field $\vec{B} = \dfrac{3\hat{r}(\vec{m}\cdot\hat{r}) - \vec{m}}{R^3}$

for aligned rotation $\vec{m} \parallel \vec{\Omega} \Rightarrow \quad \vec{r}\cdot\vec{B} = \dfrac{2(\vec{m}\cdot\hat{r})}{R^2} = \dfrac{2m\cos\theta}{R^2}$

$$\therefore \quad E_\theta(R) = -\frac{2m\Omega \sin\theta\cos\theta}{cR^2} = +\frac{m\Omega}{cR^2}\frac{d}{d\theta}\cos^2\theta = \frac{m\Omega}{3cR^2}\, 2\frac{d}{d\theta} P_2(\cos\theta)$$

$$P_2 = \frac{1}{2}(3\cos^2\theta - 1) \quad ; \quad E_\theta = -\frac{1}{r}\frac{\partial}{\partial\theta}\Phi$$

- If the outside of the NS is assumed to be vacuum then the E-field outside of the NS must be potential i.e. $\vec{\nabla}\cdot\vec{E} = 0$ & therefore it must be described by the unique potential

$$\boxed{\Phi(r,\theta) = -\frac{2m\Omega R^2}{3c\, r^3} P_2(\cos\theta) = -\frac{\Omega B R^5}{6cr^3} P_2(\cos\theta)}$$

$$\vec{E}(r,\theta) = -\vec{\nabla}\Phi = \frac{-\Omega B R^5}{12c}\left[-\frac{3(3\cos^2\theta-1)}{r^4}\,\hat{r} - \frac{6\cos\theta\sin\theta}{r^4}\,\hat{\theta}\right]$$

- The $\hat{r}$ component of the electric field at the surface of NS

$$E_r(R) \sim \frac{\Omega B R}{12c} \sim 10^7 B_{12}\Omega \text{ statVolt cm}^{-1}$$

$\boxed{1 \text{ statvolt} = 300 \text{ Volt}}$

The force on an electron $= eE \sim 5\times10^{-3} B_{12}\Omega$ dyne $= F_e$

the gravitational force on a $p^+$ at the surface –

$$F_G = \frac{GMm_p}{R^2} = 1.4\times10^{-10} \text{ dyne}$$

$$\boxed{\frac{F_e}{F_G} \approx 4\times10^7 B_{12}\Omega}$$

Note: The Coulomb force on a proton (ion) in the lattice in NS might be quite strong. If the binding energy is KeV & the separation between ions is $\sim 10^{-1}$ Å (atoms are highly elongated along the B-field), then the force on ion $\approx \frac{10^{3-12}\text{ erg}}{10^{-9}} \approx 1$ dyne! The force on $\bar{e}$s is a factor $\sim 10$ smaller; ions are harder to remove from the surface.

The strong Electric field will pull charge particles from the surface of the NS & populate the surroundings of the NS!

- Note that $\vec{E}$ is not $\perp$ $\vec{B}$ outside the NS. Therefore, charge particles will be accelerated along $\vec{B}$. The potential drop along a field line $\sim RE \approx 10^{13} B_{12}\Omega$ statvolt!
  So charge particles can be accelerated to energies of $\sim 10^{14}$ eV seen in the Crab pulsar nebula.

- The charges outside of NS will keep the plasma free of $\vec{E}$ in the comoving frame; i.e.

$$\boxed{\vec{E} = -\frac{(\vec{\Omega}\times\vec{r})\times\vec{B}}{c}}$$ within the light cylinder.

- Particle density outside NS.

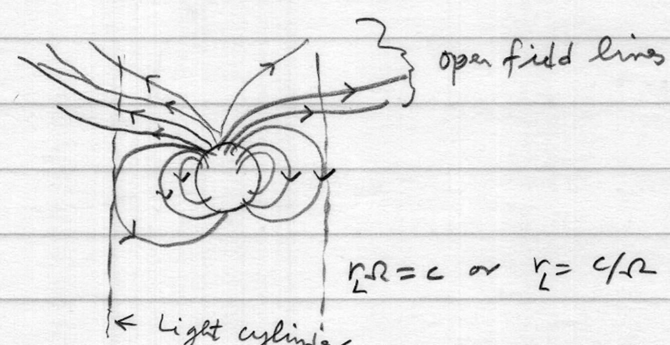
$$\vec{\nabla}\cdot\vec{E} = 4\pi\rho = 4\pi e\, n_\pm \Rightarrow n_\pm = \frac{1}{4\pi e}\vec{\nabla}\cdot\vec{E}$$

or
$$n_\pm = -\frac{1}{4\pi e c}\vec{\nabla}\cdot\left[(\vec{\Omega}\times\vec{r})\times\vec{B}\right] = -\frac{\left[\nabla\times(\vec{\Omega}\times\vec{r})\right]\cdot\vec{B}}{4\pi e c}$$

$$= -\frac{\left[\vec{\Omega}\,\vec{\nabla}\cdot\vec{r} - \vec{\Omega}\cdot\vec{\nabla}\,\vec{r}\right]\cdot\vec{B}}{4\pi e c} = -\frac{2\vec{\Omega}\cdot\vec{B}}{4\pi e c} = -\frac{2\left[3(\hat{n}\cdot\vec{\Omega})(\hat{n}\cdot\vec{m}) - (\vec{m}\cdot\vec{\Omega})\right]}{4\pi e c}$$

or
$$\boxed{n_\pm \approx \frac{\Omega B}{2\pi e c} \approx 10^{10}\, B_{12}\,\Omega \ \text{cm}^{-3}}$$

- Poynting outflow from pulsar polar cap




- Magnetic field lines can't continue to corotate with NS beyond the light cylinder, and therefore field lines will open up beyond the LC (B-lines which were closed within LC are not affected much). These open field lines carry particles & energy outward.

The energy outflow is roughly given by –

$$\frac{dE}{dt} \sim \frac{B_n^2}{8\pi}\left(\frac{R}{r_L}\right)^6 r_L^2 \approx \frac{B_n^2 R^6}{8\pi\, r_L^4} \approx \frac{B_n^2 R^6 \Omega^4}{8\pi\, c^4}$$

same as the dipole radiation formula!

## Spin-up of pulsars

Consider spherical accretion onto a magnetized NS

The spherical accretion is stopped at a place where the ram pressure of the inflow matches the magnetic pressure, and beyond this point the flow is channeled onto the magnetic poles. The equilibrium rotation of the NS is equal to the Keplerian rotation speed at the place where the ram pressure = magnetic pressure.

$$\rho V^2 \approx \frac{B^2(r)}{8\pi} \approx \frac{B^2(R)\,R^6}{8\pi\, r^6}$$

$$\rho = \frac{\dot M}{4\pi r^2 V} \quad \Rightarrow \quad \dot M V \approx \frac{B^2(R)\,R^6}{2\, r^4}$$

$$V \approx \sqrt{\frac{GM}{r}}$$

$$\therefore \quad \dot M M^{1/2} G^{1/2} \approx \frac{B^2 R^6}{2\, r^{3.5}}$$

or $\quad r_{eq} \approx 8.7\, B^{4/7} R^{12/7} \dot M^{-2/7} M^{-1/7} \approx 1.2\times 10^8\,\mathrm{cm}\; B_{12}^{4/7} R_6^{12/7} \dot m^{-2/7} m^{-\frac{3}{7}}$

$$\text{where } m = \frac{M}{1 M_\odot}, \quad \dot m = \frac{\dot M c^2}{0.1 \times \underbrace{1.3\times 10^{38}}_{\text{Eddington Luminosity for } 1 M_\odot \text{ object.}}\, m}$$

## Spin Equilibrium

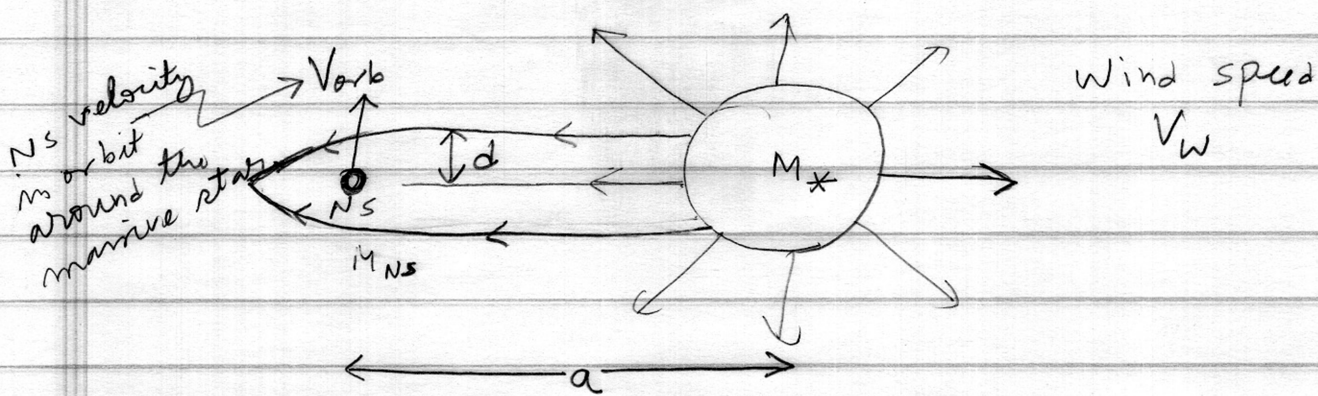
The equilibrium spin speed is the Keplerian speed at $r_{eq}$

$$\Omega_{eq} = \sqrt{\frac{GM}{r_{eq}^3}} = 8.8\ \mathrm{rad\,s^{-1}}\; B_{12}^{-6/7} R_6^{-18/7} \dot m^{3/7} m^{8/7}$$

or $\quad P_{eq} = \frac{2\pi}{\Omega_{eq}} = 2\ \mathrm{msec}\; B_9^{6/7} R_6^{18/7} \dot m^{-3/7} m^{-8/7}$

Substituting $\dot m = 1$ & $B \propto P^{1/2} \dot P^{1/2}$ we find $\boxed{P_{min}^{4/7}\, \dot P^{-3/7} \approx \text{constant}}$

## Accretion onto a NS from a companion star wind




Mass is accreted onto the NS within an impact parameter 'd' which is determined by the condition that the wind speed wrt the NS $= \sqrt{V_W^2+V_{orb}^2}$ is equal to the escape velocity from NS at distance d.

$V_{orb}$ is the speed of the NS wrt its companion

$$\frac{GM_{NS}}{d} = \frac{1}{2}\left(V_W^2+V_{orb}^2\right); \qquad d = \frac{2GM_{NS}}{V_W^2+V_{orb}^2}$$

mass loss rate in the wind $= 4\pi \rho_{wind} a^2 V_W$

The accretion rate $\dot M_{acc} = \rho_{wind}\,\pi d^2 V_{rel} = \dfrac{\dot M_W}{\cancel{4}\,a^2 V_W}\dfrac{G^2 M_{NS}^2}{\left(V_W^2+V_{orb}^2\right)^{3/2}}$

$V_{rel} = \left(V_W^2+V_{orb}^2\right)^{1/2}$

$$\text{or}\quad \frac{\dot M_{acc}}{\dot M_W} = \left(\frac{GM_{NS}}{M_{NS}+M_*}\right)^2 \frac{V_{orb}^4}{\left(V_W^2+V_{orb}^2\right)^{3/2} V_W}$$

For circular orbits $V_{orb}$, the relative speed of the two stars, is given by: $V_{orb}^2 = \dfrac{G(M_{NS}+M_*)}{a}$; where $M_*$ is the mass of the companion star & $a$ is the distance between the two.

$$\text{or}\quad \boxed{\frac{\dot M_{acc}}{\dot M_W} = \left(\frac{M_{NS}}{M_{NS}+M_*}\right)^2 \frac{\left(V_{orb}/V_W\right)^4}{\left[1+V_{orb}^2/V_W^2\right]^{3/2}}}$$

The mass loss rate for early type stars $\sim 10^{-6} M_\odot\,yr^{-1}$ (the Sun loses $3\times10^{-14} M_\odot\,yr^{-1}$). The wind speed $\sim 10^3\,km\,s^{-1}$ & $V_{orb} = \left[\dfrac{2\pi GM}{P_{orb}}\right]^{1/3} \approx 500\,km\,s^{-1}\, M_{10}^{1/3} P_{orb,1day}^{-1/3}$

$M \equiv M_{NS}+M_*$; $P_{orb}$: orbital period

$\therefore\ \dot M_{acc} \approx 7\times10^{-4}\dot M_W \sim 7\times10^{-10} M_\odot\,yr^{-1} = 4\times10^{16}\,g\,s^{-1} \rightarrow$ luminosity $\approx 4\times10^{36}\,erg\,s^{-1}$!

# Bondi Accretion

## Spherically symmetric case in steady state

mass conservation $\quad \frac{\partial}{\partial t}\rho + \nabla\cdot(\rho\vec{v}) = 0$

steady state $\Rightarrow \frac{\partial \rho}{\partial t} = 0 \quad \therefore \nabla\cdot(\rho\vec{v}) = \frac{\partial}{r^2\partial r}(r^2\rho v) = 0$

or $\quad r^2\rho v = \text{constant}$

$$4\pi\rho r^2 v = \dot{M} \qquad (1)$$

Momentum Equation or Euler's equation

$$\rho\,\vec{v}\cdot\vec{\nabla}\vec{v} = -\nabla p - \rho\nabla\Phi$$

or $\quad \frac{d}{dr}\frac{v^2}{2} + \frac{1}{\rho}\frac{dp}{dr} + \frac{d\Phi}{dr} = 0 \qquad (2)$

for polytropic equation of state; $p \propto \rho^\Gamma$ or $\frac{p}{p_\infty} = \frac{\rho^\Gamma}{\rho_\infty^\Gamma}$
(or adiabatic flow)

$$\frac{d}{dr}\left(\frac{v^2}{2} + \frac{\Gamma p}{(\Gamma-1)\rho} + \Phi\right) = 0$$

or $\quad \frac{v^2}{2} + \frac{c_s^2}{\Gamma-1} - \frac{GM}{r} = \text{constant} = \frac{c_\infty^2}{\Gamma-1} \qquad (2a)$

$$c_s^2 = \Gamma\frac{p}{\rho} = \Gamma\frac{p_\infty}{\rho_\infty}\left(\frac{\rho}{\rho_\infty}\right)^{\Gamma-1} = c_\infty^2\left(\frac{\rho}{\rho_\infty}\right)^{\Gamma-1} \qquad (2b)$$

## Transonic solution

We can rewrite eq. (1) & (2) as –

$$\frac{d\ln\rho}{dr} + 2\frac{d\ln r}{dr} + \frac{d\ln v}{dr} = 0 \qquad (3)$$

$$\frac{dv^2}{dr} + 2c_s^2\frac{d\ln\rho}{dr} + \frac{2GM}{r^2} = 0 \qquad (4)$$

Eliminating $\ln\rho$ – $\quad \frac{dv^2}{dr} - \frac{c_s^2}{v^2}\frac{dv^2}{dr} + \frac{2GM}{r^2} - \frac{4c_s^2}{r} = 0$

or $$\frac{dv^2}{dr} = \frac{\frac{2v^2}{r}\left(2c_s^2 - GM/r\right)}{\left(v^2 - c_s^2\right)} \qquad (5)$$

At large $r$ (in the limit $r \to \infty$) the numerator is +ve & the denominator is -ve (as $v \to \infty$ & $c_s \to c_\infty$). Therefore, $v^2$ increases with decreasing $r$. At small $r$, the numerator is -ve. If we want a solution such that $v$ increases monotonically with decreasing $r$, then we must have a +ve denominator i.e $v > c_s$ i.e. the flow is supersonic at small $r$. Since the denominator is '0' at the sonic point ($v = c_s$), the numerator too must vanish here to keep the solution regular i.e.

at the sonic point – $$c_s^2 = \frac{GM}{2r} \qquad (6)$$

Substituting this back into (2a) we find –

$$\frac{c_s^2}{2} + \frac{c_s^2}{\Gamma - 1} - 2c_s^2 = \frac{c_\infty^2}{\Gamma - 1} \quad \text{or} \quad \frac{5 - 3\Gamma}{2}\, c_s^2 = c_\infty^2 \qquad (7)$$

The accretion rate $\dot{M} = 4\pi \rho_s r_s^2 v_s = \pi \frac{(GM)^2}{c_s^3(r_s)} \rho_\infty \left(\frac{c_s}{c_\infty}\right)^{\frac{2}{\Gamma - 1}}$

(subscript 's' is for sonic point)

or $$\boxed{\dot{M} = \frac{\pi (GM)^2 \rho_\infty}{c_\infty^3} \left(\frac{5 - 3\Gamma}{2}\right)^{\frac{3\Gamma - 5}{\Gamma - 1}}} \qquad (8)$$

Note: we can understand this result in a simple way – the rate above is what we obtain at the radius where the "free fall" starts i.e. $\frac{GM}{r} \approx c_\infty^2 \Rightarrow \dot{M} \sim \rho_\infty r^2 c_\infty \sim \rho_\infty (GM)^2 / c_\infty^3$.

For $M \sim 1 M_\odot$, $\rho_\infty \sim 1\,cc$ & $c_\infty \sim 10\,km\,s^{-1}$ $\dot{M} \sim 10^9\,g\,s^{-1}$ which results in a luminosity of $\sim 10^{29}\,erg\,s^{-1}$ if accreted onto a NS.

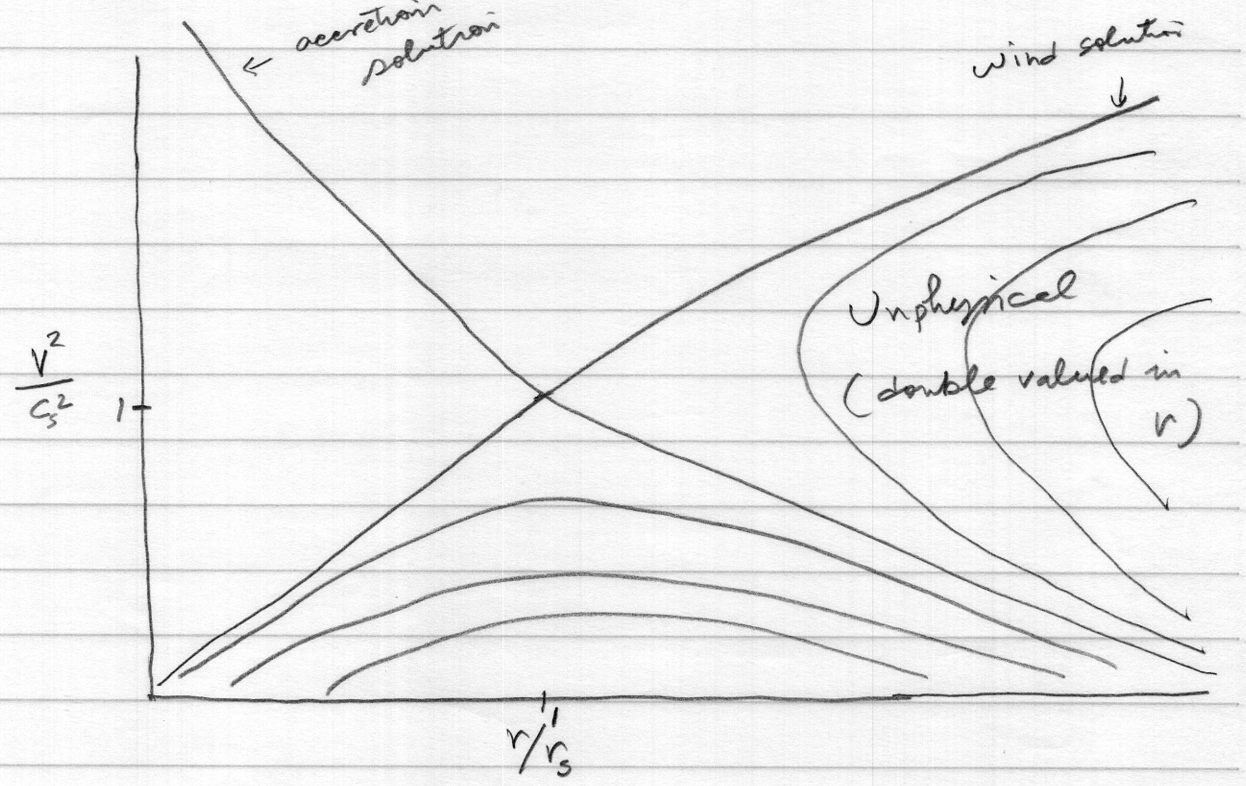
Note 1: The transonic accretion rate given by eq. (8) is the maximum possible accretion in a steady state. For smaller accretion rate the speed's subsonic everywhere & vanishes as $r \to 0$. (See fig. below)

2: Solar/stellar thermal wind is also described by the same set of equations i.e. 1 & 2.

3: The accretion rate when the star moves w.r.t. ISM with speed $V$ is: $\dot{M} \sim \frac{\rho_\infty (GM)^2}{(V^2 + c_\infty^2)^{3/2}}$

4: Behavior at small $r$: For $r \ll r_s$ we can neglect the RHS of (2a) since $c_s \gg c_\infty$ & balancing the +ve & negative terms on the LHS we find $v^2 \sim \frac{GM}{r}$ ($c_s < v$ for supersonic flow) & the mass conservation $\Rightarrow \rho \propto r^{-2} v^{-1} \propto r^{-3/2}$.

5: There are no steady state transonic solutions for $\Gamma > 5/3$.



# Roche Analysis

## Potential in a rotating frame

The equation of motion in a rotating frame —

$$\frac{d\vec{v}}{dt} = \frac{\partial \vec{v}}{\partial t} + \vec{v}\cdot\vec{\nabla}\vec{v} = -\nabla\Phi - \underbrace{2\vec{\Omega}\times\vec{v}}_{\text{coriolis}} - \underbrace{\vec{\Omega}\times(\vec{\Omega}\times\vec{r})}_{\text{centrifugal force}}$$

For two point masses —

$$\Phi = -\frac{GM_1}{|\vec{r}-\vec{r}_1|} - \frac{GM_2}{|\vec{r}-\vec{r}_2|}$$

$\vec{r}_1$ & $\vec{r}_2$ are measured from the CM.

$$\vec{\Omega}\times(\vec{\Omega}\times\vec{r}) = (\vec{\Omega}\cdot\vec{r})\vec{\Omega} - \Omega^2\vec{r} \equiv -\Omega^2\tilde{\omega} = -\vec{\nabla}\left(\tfrac{1}{2}\Omega^2\tilde{\omega}^2\right)$$

cylindrical polar coordinate distance i.e. distance from CM in orbital plane.

$$\therefore\ \frac{d\vec{v}}{dt} = -\nabla\left[\Phi - \tfrac{1}{2}\Omega^2\tilde{\omega}^2\right] - 2\vec{\Omega}\times\vec{v}$$

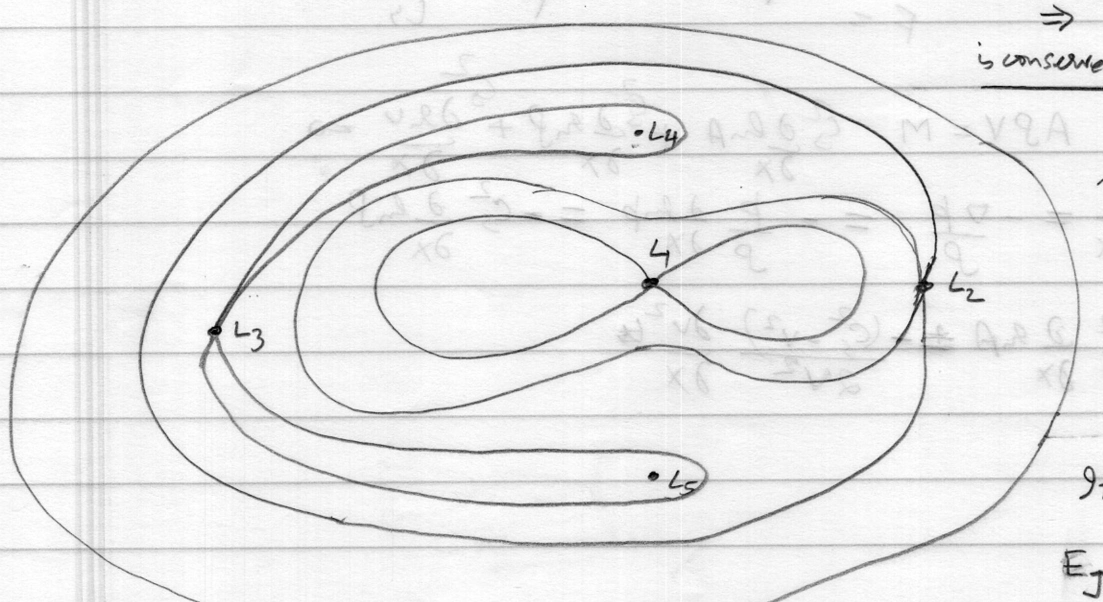
multiply with $\vec{v}$

Roche Potential $\boxed{\Phi_R \equiv \Phi - \tfrac{1}{2}\Omega^2\tilde{\omega}^2}$

$$\frac{d\,v^2/2}{dt} = -\vec{v}\cdot\vec{\nabla}\Phi_R = -\frac{d}{dt}\Phi_R$$

$$\Rightarrow \frac{v^2}{2} + \Phi - \frac{\Omega^2\tilde{\omega}^2}{2} = E_J$$

is conserved ($E_J$ is Jacobian integral).




$L_1$, $L_2$ & $L_3$ are saddle points & are unstable.

$L_4$ & $L_5$ are maxima but stable due to coriolis force.

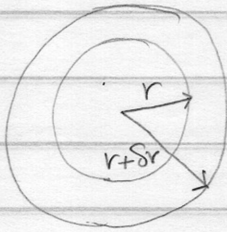
It can be shown that (see Binney & Tremaine)

$$E_J = E - \vec{\Omega}\cdot\vec{L}$$

$E$: energy in inertial frame; $\vec{L}$: angular momentum

# Accretion disk

## 1. Order of magnitude estimates




Let the accretion rate be $\dot{M}$. In steady state $\dot{M}\delta t$ mass will pass through radius $r+\delta r$ & $r$ in time $\delta t$. The amount of energy released in time $\delta t$ between these radii

$$= \delta r \frac{d}{dr}\left(-\frac{GM}{2r}\dot{M}\delta t\right) = + \frac{GM\dot{M}}{2r^2}\delta r\,\delta t \qquad (1)$$

If this energy release, or thermal energy, is radiated away in time $\delta t$ in the area between $r$ & $r+\delta r$ as black body radiation then –

$$\sigma T^4 \underbrace{(2\pi r\delta r)}_{\text{area}} \delta t = \frac{GM\dot{M}}{2r^2}\delta r\,\delta t$$

or

$$\boxed{T = \left(\frac{GM\dot{M}}{4\pi\sigma}\right)^{\frac{1}{4}} r^{-3/4}} \qquad (2)$$

For $M = 1M_\odot$, $\dot{M} = \dot{M}_{Edd}$, $r = 100\,\text{km}$, $T = 4\times10^6\,\text{K}$.

**Observed Spectrum:**

Let us say that we make observation at frequency $\nu$. The part of the disk that will contribute to the flux at $\nu$ is upto a radius $r_\nu$ such that – $kT(r_\nu) = h\nu$ (for $r > r_\nu$ the contribution falls of exponentially). Or $r_\nu \propto \nu^{-4/3}$.

The observed flux $f_\nu = 2\pi\int_0^{r_\nu} dr\, r\, \frac{2\nu^2 kT(r)}{c^2} \propto \nu^2 r_\nu^{5/4}$

or $\boxed{f_\nu \propto \nu^{1/3}}$

{ the exact integral can also be done easily:
$f_\nu = 2\pi\int_0^\infty dr\, r \frac{2\nu^3}{c^2[e^{h\nu/kT}-1]} \propto \nu^{3-\frac{8}{3}} \times \int \frac{d(r\nu^{4/4})(r\nu^{4/3})}{(e^{h\nu r^{3/4}}-1)}$

Dwarf nova have this spectrum during outburst (but not during quiescent period – probably because the disk is optically thin during quiet period).

# Structure of Thin Accretion disk

1. Mass conservation

The rate of mass inflow should be r-independant i.e.

radial velocity (inward)

$$2\pi r \Sigma V_r = \dot{M} = \text{constant} \qquad (1)$$

surface mass density $\equiv \int_{-\infty}^{\infty} dz\, \rho$

2. Angular momentum conservation

a. Inward flux of angular momentum carried by $\dot{M}$

$$\mathcal{L}_{In} = \dot{M}(GMr)^{1/2} \qquad (2a)$$

b. Outward flux due to viscous stress

Let G be the torque associated with viscous stress that the disk within r exerts on the exterior disk. Then the rate of angular momentum transferred from the inner to the outer disk, i.e. the angular momentum flux, is G.

$G - \mathcal{L}_{in}$ the net outward transport of angular momentum which should be r indenpant i.e.

$$G - \dot{M}(GMr)^{1/2} = \text{consotat}$$

if the torque vanishes at the inner edge of disk at $r_{min}$ e.g. for a BH then -

$$G = \dot{M}\left(r^{1/2} - r_{min}^{1/2}\right)(GM)^{1/2} \qquad (2)$$

c. For local viscosity & shear proportional to the velocity gravient, the torque

$$G = -2\pi r \int dz\, r\, \rho \nu\, r \frac{d\Omega}{dr}$$

shear stress i.e. force per unit area

Kinamatricviscosity

We neglect pressure gradient in the r-direction, & therefore $\Omega$ has Keplerian velocity profile i.e. $\frac{d\Omega}{dr} = -\frac{3}{2}\frac{\Omega}{r} = -\frac{3}{2r}\sqrt{\frac{GM}{r^3}}$

$\therefore \quad G = 3\pi \Sigma \nu \sqrt{GMr} \approx \dot{M} \sqrt{GMr}$

[using eq. (2) for $r \gg r_{min}$

or $\dot{M} = 3\pi \Sigma \nu$ (3)

The viscosity $\nu$ must be much greater than molecular viscosity to drive accretion at a rate needed to explain the luminosity observed for x-ray binary systems or AGN disks. Shakura & Sunyaev suggested that the viscosity in the disk is turbulent viscosity –

$$\nu = \alpha \, c_s H \tag{4}$$

$\alpha$ – parameter; $c_s$ – sound speed; $H$ – vertical scaleheight

(Note: ordinary molecular viscosity is $\sim c_s \lambda$; where $\lambda$ is mean free path length for particles).

3. Energy conservation

a. Gravitational energy release by accreting gas between $r$ & $r+\delta r$

$$\delta E = -\dot{M} \frac{d}{dr}\left(\frac{GM}{2r}\right) \delta r = \frac{\dot{M} GM \delta r}{2r^2}$$

$$\frac{dE}{dr} = \frac{GM\dot{M}}{2r^2} \tag{5a}$$

b. The work done by the viscous torque on the ring between

$r$ & $r+\delta r$ $= -\dfrac{dG\Omega}{dr} \delta r$

the contribution to the energy release by viscous torque $= -\dfrac{dG\Omega}{dr}$ (5b)

Adding the above two contributions –

$$\frac{dE}{dr} = \frac{GM\dot{M}}{2r^2} - \frac{d}{dr} G\Omega \tag{5}$$

Making use of eq. (2) for G into (5) –

$$\frac{dE}{dr} = \frac{GM\dot{M}}{2r^2} - \frac{d}{dr}\left\{\dot{M}(GM)^{1/2}\left[r^{1/2} - r_{min}^{1/2}\right]\sqrt{\frac{GM}{r^3}}\right\}$$

$$= \frac{GM\dot{M}}{2r^2} - \dot{M}GM\left\{-\frac{1}{r^2} + \frac{3}{2}\frac{r_{min}^{1/2}}{r^{5/2}}\right\}$$

$$\frac{dE}{dr} = \frac{3GM\dot{m}}{2r^2}\left[1 - \left(\frac{r_{min}}{r}\right)^{1/2}\right] \qquad (6)$$

Integrating this over $r$ – from $r_{min}$ to $\infty$ – we find that the total energy release is $= \frac{G\dot{M}M}{2r_{min}}$ as expected. For $r \gg r_{min}$ $\frac{dE}{dr} = \frac{3GM\dot{M}}{2r^2}$. Therefore, twice as much energy is released as a result of viscous torque as the decrease in the PE of infalling gas!

The energy release per unit area $Q^+ = \frac{1}{2\pi r}\frac{dE}{dr}$

$$\text{or} \quad Q^+ = \frac{3GM\dot{m}}{4\pi r^3}\left[1 - \left(\frac{r_{min}}{r}\right)^{1/2}\right] = \frac{3\dot{M}\Omega^2}{4\pi}\left[1 - \left(\frac{r_{min}}{r}\right)^{1/2}\right] \qquad (7)$$

4. Radiation

For optically thick disk the energy loss rate / area –

$$Q^- \sim \frac{8acT^4}{3\tau} \qquad (8)$$

; $T$ is temperature in disk mid plane

where $\tau = \kappa\Sigma$ is the optical depth of the disk in vertical direction; $\kappa \sim 0.4\ cm^2 g^{-1}$ when opacity is dominated by the Thomson scattering & $\kappa \sim 7\times10^{22}\rho T^{-3.5} cm^2 g^{-1}$ for Rosseland opacity which is a combination of bound free & free-free opacity for solar abundance.

## 5. Vertical disk structure

$$\frac{dp}{dz} = -\rho g = -\rho \Omega^2 z$$

for an isothermal structure in the vertical direction

The scale height $H = c_s/\Omega$ (8)

& $p = p_0 \, e^{-\frac{\Omega^2 z^2}{2c_s^2}}$ (9)

$$p = \frac{1}{3} a T^4 + \frac{\rho k T}{\mu m_p} \qquad (10)$$

$\mu$ ← mean molecular weight

$a = 7.5\times10^{-15}$ erg cm$^{-3}$ K$^{-1}$ is the radiation constant

## Disk Structure in steady state

We will assume that heating rate is equal to radiative energy loss rate at all r locally (this does not have to be so).
The equations for thin disk, (optically thick), are:

$$\dot{M} = 2\pi r \Sigma V_r \qquad (1)$$

$$\dot{M} = 3\pi \Sigma \nu \qquad (3)$$

$$\nu = \alpha c_s H = \alpha c_s^2/\Omega \qquad (4)$$

$$Q^+ = Q^- \text{ or } \frac{3\dot{M}\Omega^2}{4\pi}\eta = \frac{8acT^4}{3\kappa\Sigma} \qquad (7)\ \&\ (8)$$

where $\eta = 1 - \sqrt{\frac{r_{min}}{r}}$

We have 4 equations with 4 unknowns viz. $\Sigma, V_r, T$ & $\nu$ & can solve these easily to determine $\Sigma, T$ & $V_r$ as a function of r.

Let us specialize to the case where the pressure is dominated by the gas pressure & $\mathcal{K}$ by Thomson scattering. In this case $p = \frac{\rho k T}{m_p \mu}$ & $c_s^2 = \frac{kT}{m_p \mu}$. The disk equations become:

$$\dot{M} = 2\pi r V_r \Sigma \qquad (11)$$

$$\dot{M} = 3\pi \Sigma \alpha c_s^2/\Omega = \frac{3\pi k \alpha \Sigma T}{m_p \mu \Omega} \qquad (12)$$

$$\frac{3\dot{M}\Omega^2\eta}{4\pi} = \frac{8acT^4}{3\mathcal{K}\Sigma} = \frac{8acT^4}{3\times 0.4\Sigma} \qquad (13)$$

$\eta \equiv 1-(r_{min}/r)^{1/2}$

Eliminating $\Sigma$ from (12) & (13) —

$$\frac{3\dot{M}\Omega^2\eta}{4\pi} = \frac{8acT^5\, 3\pi k\alpha}{1.2\,\dot{M}}$$

mass accretion rate in units of Eddington rate.

$$\text{or } T = \left[\frac{1.2\,\dot{M}^2\Omega^2\eta}{32\pi^2 ack\alpha}\right]^{1/5} \sim 10^{11}\text{K}\; \dot{m}^{2/5}\left(\frac{M}{M_\odot}\right)^{3/5} r_7^{-3/5}\,\alpha^{-1/5} \qquad (14a)$$

Note: $T \propto r^{-3/5}$, NOT $T\propto r^{-3/4}$ derived earlier by balancing energy generation & loss rates, and the reason for this is that $T$ given by (14a) is the temperature in disk midplane where as $T\propto r^{-3/4}$ refers to the temperature above the disk mid-plane at the "photosphere".

Substituting (14a) into (12) we find $\rho = \frac{\Sigma}{H} = \frac{\Sigma\Omega}{c_s} \propto r^{-\frac{21}{10}}$ & from (11) $V_r \propto r^{-1/10}$.