Magnetic dipole radiation formula

Energy loss rate due to magnetic dipole radiation -

\[
\frac{dE}{dt} = - \frac{2}{3} \frac{\dot{m}^2}{c^3}
\]

(where \( \dot{m} \) is magnetic dipole moment,

the magnetic field in terms of \( \dot{m} \)

given by \( B = \frac{3 \dot{r}(r, \dot{m})}{r^3} \)

[See Jackson 5.5b]

\( \dot{m} \sim \frac{B R^3}{c} ; B \sim \text{mag field} \)

\( \text{at NS surface} \)

\& \( R \) is NS star radius. \( \Rightarrow \)

\( \dot{m} \sim m^2 \sin^2 \theta \) \( \theta \) is the angle between \( \vec{r} \) \& \( \dot{m} \).

\[
\frac{dE}{dt} = - \frac{2}{3} \frac{m^2 \dot{r}^2 \sin^2 \theta}{c^3} = - \frac{1}{6} \frac{B^2 R^6 \dot{r}^2 \sin^2 \theta}{c^3}
\]

\( E = \frac{1}{2} I \dot{\omega}^2 \)

\[
\omega \frac{d\omega}{dt} = - \frac{B^2 R^6 \dot{r}^2 \sin^2 \theta}{6 I c^3}
\]

Observationally it is found that \( \frac{d\omega}{dt} \propto \omega^2 \) with \( \omega' \)
as small as 1.4 for some pulsar \& \( \sim 2.8 \) for some others.
(\( \dot{\omega} \) suggests that there is a torque in NS due to some process

other than mag dipole radiation. It could be for instance dust

particle outflow.)

Take \( \frac{d\omega}{dt} = - K \dot{\omega} \Rightarrow \frac{1}{\omega^{a-1}(t)} = \frac{1}{\omega^{a-1}(t_0)} = K(a-1)(t-t_0) \)

for \( \omega(t) \ll \omega(t_0) \), the pulsar spin down age -

\[
(t-t_0) = \frac{\omega}{K\dot{\omega}^{a-1}(t)(a-1)} = \frac{\omega}{\dot{\omega}^{a-1}(t)(a-1)}
\]

Turns out to be OK for many NS & historical SNR pulsars.
Pulsar distance determination (dispersion measure)

\[ \omega^2 = \omega_p^2 + k^2 c^2 \]

\[ \omega_p^2 = \frac{4 \pi n_e e^2}{m_e} = 3 \times 10^9 n_e \quad \text{\(\gamma_p = \frac{\omega_p}{2\pi} = 8.5 n_e^{1/2} \text{kHz}\)} \]

Time for a pulse to arrive

\[ t_{\text{obs}}(\omega) = \int_0^d \frac{d}{V_g} \quad V_g = \frac{d\omega}{dk} = \frac{k e^2}{\omega} = \frac{c}{\omega} \sqrt{\omega^2 - \omega_p^2} = c \frac{\sqrt{1 - \omega_p^2/\omega^2}}{2\omega^2} \]

\[ t_{\text{obs}}(\omega) = \frac{d}{c} + \frac{1}{2c^2} \int d\omega \omega^2 = \frac{d}{c} + \frac{4\pi n_e^2}{cm_e^2} \int d\omega n_e \]

\[ \gamma \quad t_{\text{obs}}(\omega) = \frac{d}{c} + \frac{4\pi n_e^2}{cm_e^2} \langle n_e \rangle d \]

\[ \frac{dt_{\text{obs}}}{d\omega} = - \frac{4\pi n_e^2}{cm_e^2} \langle n_e \rangle d \]

or

\[ d = \frac{cm_e^2}{4\pi n_e^2 \left| \frac{dt_{\text{obs}}}{d\omega} \right|} \frac{1}{\langle n_e \rangle} \]

\[ \langle n_e \rangle \approx 0.03 \text{ cm}^{-3} \text{ in the Galaxy; but it varies along different lines of sight & different parts of the Galaxy.} \] 

\[ \langle n_e \rangle \text{ is determined from pulsars for which we know the distance in an independent way.} \]
Pulsar Magnetosphere

First calculate scale height for NS and show that the particle density is vanishingly small just a few meters from the surface of a NS.

Goldreich-Julian Model

The interior of the NS is highly conducting fluid so there should be no E-field in its comoving frame (current $\vec{j} = 6 \vec{E}$; $\vec{j}$ is so large that a small $\vec{E}$ will cause a huge current).

There must be an $\vec{E}$ field in the NS as seen by an inertial frame observer; call this field $\vec{E}$. Since the field in the comoving frame is zero:

$$\Rightarrow \quad \vec{E} + \frac{\vec{v} \times \vec{B}}{c} = 0 \quad \vec{v} = \frac{\vec{r} \times \vec{r}}{c}$$

$$\Rightarrow \quad \vec{E} = -\frac{(\vec{r} \times \vec{r}) \times \vec{B}}{c} = \frac{(\vec{B} \cdot \vec{r}) \vec{r} - (\vec{B} \cdot \vec{r}) \vec{r}}{c}$$

The $\theta$-component of the $\vec{E}$ at the surface of the NS is:

$$E_{\theta}(R) = -\frac{(\vec{B} \cdot \vec{r})}{c} \Omega \sin \theta \quad \text{in dipole magnetic field } \vec{B} = \frac{3 \vec{r} (\vec{r} \cdot \vec{r})}{R^3} - \frac{\vec{m}}{R^3}$$

for aligned rotation $\vec{m} \parallel \vec{r}$

$$\Rightarrow \quad \vec{r} \cdot \vec{B} = \frac{2(\vec{r} \cdot \vec{r})}{R^2} = \frac{2m \cos \theta}{R^2}$$

$$\Rightarrow \quad E_{\theta}(R) = -\frac{2m}{cR^2} \sin \theta \cos \theta = \frac{m}{cR^2} \frac{d}{d \theta} \left[\sin^2 \theta \right]$$

$$P_2 = \frac{1}{2} (3\cos^2 \theta - 1) \quad \Rightarrow \quad E_{\theta} = -\frac{1}{r^2} \frac{d}{d \theta}$$

E-field outside of the NS must be potential i.e. $\nabla \cdot \vec{E} = 0$ & therefore it must be described by a unique potential

$$\Phi(r, \theta) = -\frac{2m}{cR^2} \frac{R^2}{3c} \frac{d}{d \theta} P_2(\cos \theta) = -\frac{1}{cR^2} \frac{R^2}{3c} \frac{d}{d \theta}$$

$$= -\frac{1}{6cR^3}$$
\[ E(r, \theta) = -\nabla \mathbf{E} = -\frac{\alpha B R^2}{12c} \left[ \frac{3(\cos^2 \theta - 1)}{r^4} \hat{r} - \frac{6 \cos \theta \sin \theta \hat{\theta}}{r^4} \right] \]

- The \( \hat{r} \) component of the electric field at the surface of NS is

\[ E_r(R) \sim \frac{eBR}{2c} \sim 10^7 \text{ B}_{12} \text{ Q} \text{ V}\text{stat} \text{ cm}^{-1} \]

- The force on an electron in the NS is

\[ F_e \sim eE \sim 5 \times 10^3 \text{ B}_{12} \text{ Q} \text{ dyn cm}^{-2} \]

- The gravitational force on a proton at the surface is

\[ F_g = \frac{Gm_m m_p}{R^2} = 1.4 \times 10^{-10} \text{ dyn cm}^{-2} \]

- The force on a proton in the NS is

\[ F_p \sim 4 \times 10^7 \text{ B}_{12} \text{ Q} \]

- The strong electric field will pull charge particles from the surface of the NS and populate the surroundings of the NS!

Note: The Coulomb force on a proton in the lattice in the NS might be quite strong.

If the binding energy is \( E_{\text{bind}} \approx 10^{-12} \text{ eV} \) (when ions are highly aligned along the B-field), then the force on an ion is \( 10^{-12} \text{ eV cm}^{-2} \approx 1 \text{ dyn cm}^{-2} \). This force is a factor 10^{-10} smaller than the force on an electron.

- Remove from its surface.

- The charge outside of the NS will keep the plasma free of the electric field, i.e.

\[ E' = -\left( \frac{2 \times 2}{c} \right) \mathbf{B} \]

within the light cylinder.
Particle density outside NS.

\[ \nabla \cdot \mathbf{E} = 4\pi \rho = 4\pi e n_+ \Rightarrow n_+ = \frac{1}{4\pi e} \nabla \cdot \mathbf{E} \]

\[ n_+ = -\frac{1}{4\pi e} \nabla \cdot [ (\mathbf{E} \times \mathbf{B}) \times \mathbf{E} ] = -\frac{[\nabla \times (\mathbf{E} \times \mathbf{B})] \cdot \mathbf{B}}{4\pi e c} \]

\[ = \frac{[\nabla \cdot (\mathbf{B} \cdot \mathbf{E}) - \mathbf{E} \cdot \nabla \cdot \mathbf{B}] \cdot \mathbf{B}}{4\pi e c} = -\frac{\mathbf{B} \cdot \mathbf{B}}{4\pi e c} = -\frac{\mathbf{B} \cdot \mathbf{B}}{4\pi e c} \]

\[ n_+ \approx \frac{\alpha B}{4\pi e} \approx 10^{10} \text{ B}_{12} \alpha \text{ cm}^{-3} \]

Poynting outflow from pulsar polar cap

Magnetic field lines can't continue to contact with NS beyond the light cylinder, and therefore field lines will open up beyond the LC. (B-lines which were closed within LC are not affected much). These open field lines carry particles & energy outward.

\[ \text{S} \text{e}_0 \text{ e}^{-} \text{g}_{\text{n}} \text{ outflow is roughly given by} \]

\[ \frac{dE}{dt} \approx \frac{B^2 (B_0^2 \beta^2)}{8\pi} \frac{r^2}{L^4} \approx \frac{E_B R_0^6}{9\pi L^4} \approx \frac{B_0^2 R_0^6 \alpha^4}{8\pi C^4} \]

Same as the dipole radiation formula!
Spin-up of pulsars

Consider spherical accretion onto a magnetized NS

The spherical accretion is stopped at a place where the ram pressure of the inflow matches the magnetic pressure, and beyond this point the flow is channeled onto the magnetic poles. The equilibrium rotation of the NS is equal to the Keplerian rotation speed at this place where the ram pressure = magnetic pressure.

\[ PV^2 = \frac{B^2 r^2}{8\pi} \approx \frac{B^3 R^6}{8\pi r^6} \]

\[ \frac{P}{4\pi r^2 V} = \frac{M}{6\pi r^2 V} \Rightarrow \frac{M V}{r} \approx \frac{B^3 R^6}{2 r^4} \]

\[ V \approx \sqrt{\frac{GM}{r}} \]

\[ \frac{M}{M_\odot} \frac{V}{c} \approx \frac{B^2 R^6}{2 r^{3.5}} \]

\[ R_{eq} \approx 8.7 B^{4/7} R^{12/7} M^{-1/7} \approx 7.2 \times 10^8 \text{ cm} \quad B_{12} \quad R_6 \quad M^{-1/7} \]

where \( m = \frac{M}{M_\odot} \), \( \dot{m} = \frac{\dot{M} c^2}{0.1 \times 1.3 \times 10^{38} \text{ m}} \)

Eddington luminosity for 1 M\odot object.

Spin Equilibrium

The equilibrium spin speed is the Keplerian speed at \( R_{eq} \)

\[ \Omega_{eq} = \sqrt{\frac{GM}{R_{eq}^2}} = 8.8 \times 10^{-6} \quad B_{12} \quad R_6 \quad M^{-1/7} \quad m^{-3/7} \quad \text{rad s}^{-1} \]

\[ \Omega_{eq} = \frac{\Omega_{Kep}}{R_{eq}} \approx 2 \text{ M}\odot \quad B_{12} \quad R_6 \quad M^{-1/7} \quad m^{-3/7} \quad \text{rad s}^{-1} \]

Substituting \( m = 1 \) & \( B = 8 \times 10^{12} \) we find \( \Omega_{Kep} \approx 9/4 \)
Accretion onto a NS from a companion star wind

\[ \text{NS velocity} \rightarrow \text{V}_{\text{orb}} \]
\[ \text{NS orbit} \rightarrow \text{massive star} \]
\[ \text{M}_\text{NS} \]
\[ \text{W} \]
\[ \text{V}_{\text{w}} \]
\[ \text{a} \]

Mass is accreted onto the NS within an impact parameter \( d \) which is determined by the condition that the wind speed \( V_{\text{w}} \) at the NS surface \( V_{\text{w}} \) is equal to the escape velocity \( V_{\text{esc}} \) at distance \( d \):

\[ V_{\text{esc}} = \frac{1}{2} \left( \frac{GM_{\text{NS}}}{d^2} \right)^{1/2} \]

The escape velocity is:

\[ V_{\text{esc}} = \frac{V_{\text{w}}^2 + V_{\text{orb}}^2}{V_{\text{w}} + V_{\text{orb}}} \]

The accretion rate \( M_{\text{acc}} \) is given by:

\[ M_{\text{acc}} = \pi \text{wind} \pi a^2 V_{\text{w}} d \frac{G(M_{\text{NS}} + M_\star)}{a^2 V_{\text{w}} \left( V_{\text{w}}^2 + V_{\text{orb}}^2 \right)^{3/2}} \]

For circular orbits \( V_{\text{orb}} \), the relative speed of the two stars \( \frac{V_{\text{w}}}{V_{\text{orb}}} \) is given by:

\[ \frac{V_{\text{w}}}{V_{\text{orb}}} = \left( \frac{M_{\text{NS}}}{M_\star} \right)^{1/2} \left( \frac{V_{\text{w}}}{V_{\text{orb}}} \right)^{3/2} \]

The mass accretion rate for early type stars \( \approx 10^{-6} \text{M}_\odot \text{yr}^{-1} \) (for sun \( \approx 3 \times 10^{-14} \text{M}_\odot \text{yr}^{-1} \)). The wind speed \( \approx 10^3 \text{km} \text{s}^{-1} \) & \( V_{\text{orb}} = \frac{1}{2} \left( \frac{GM_{\text{NS}}}{a^2} \right)^{1/2} \).

\[ \frac{M_{\text{acc}}}{M_\star} = \left( \frac{M_{\text{NS}}}{M_\star} \right)^{1/2} \left( \frac{V_{\text{orb}}}{V_{\text{w}}} \right)^{4/3} \left[ 1 + \frac{V_{\text{w}}}{V_{\text{orb}}} \right]^{-3/2} \]

The mass accretion rate for early type stars \( \approx 7 \times 10^{-4} \text{M}_\odot \text{yr}^{-1} \) & \( \text{luminosity} \approx 4 \times 10^{36} \text{erg} \text{s}^{-1} \).
Bondi Accretion

Spherically symmetric case in steady stat

mass conservation \[ \frac{\partial P + \nabla \cdot (P \mathbf{v}^2)}{\partial t} = 0 \]

steady state \[ \frac{\partial P}{\partial t} = 0 \quad \Rightarrow \quad \nabla \cdot (P \mathbf{v}^2) = \frac{2}{r} (r^2 P \mathbf{v}) = 0 \]

or \[ r^2 P \mathbf{v} \text{ constant} \]

\[ 4\pi r^2 P \mathbf{v} = \dot{M} = \text{constant} \quad \text{(1)} \]

Momentum equation or Euler's equation

\[ P \mathbf{v} \cdot \nabla \mathbf{v} = -\nabla P - \rho \nabla \Phi \]

or \[ \frac{d}{dr} \left( \frac{r^2}{2} \right) + \frac{1}{\rho} \frac{dp}{dr} + \frac{d\Phi}{dr} = 0 \quad \text{(2)} \]

for polytropic equation of state; \( \rho \propto \rho^p \) or \( \frac{P}{P_0} = \frac{\rho}{\rho_0}^p \)

(or adiabatic flow)

\[ \frac{d}{dr} \left( \frac{v^2}{2} + \frac{\Gamma P}{(\Gamma - 1)P} \right) = 0 \]

\[ \frac{\Gamma}{\Gamma - 1} \left( \frac{v^2}{2} + \frac{c_s^2}{\rho} \right) = \text{constant} = \frac{c_{s,0}^2}{\Gamma - 1} \quad \text{(2a)} \]

\[ c_s^2 = \frac{P}{\rho} = \frac{1}{\rho_0} \frac{P}{P_0} (\frac{\rho}{\rho_0})^{\Gamma - 1} = \frac{c_{s,0}^2}{\rho_0} (\frac{\rho}{\rho_0})^{\Gamma - 1} \quad \text{(2b)} \]

Transonic solution

We can rewrite eq. (1) & (2) as:

\[ \frac{d}{dr} \left( \frac{r^2}{2} \right) + \frac{1}{\rho} \frac{dp}{dr} = \frac{d\Phi}{dr} = 0 \]

\[ \frac{d^2 v^2}{dr^2} + 2 \frac{c_s^2}{\rho} \frac{dv^2}{dr} + \frac{2GM}{r^2} = 0 \quad \text{(4)} \]

Eliminate \[ \frac{dv^2}{dr} = \frac{c_{s,0}^2}{\rho_0} \frac{2GM}{r^2} - 4 \frac{c_s^2}{P} \]
\[ \frac{dv^2}{dr} = \frac{2v^2(2\xi^2 - 6\xi)}{(v^2 - \xi^2)} \quad (5) \]

At large \( r \) (in the limit \( r \to \infty \)) the numerator is +ve & the denominator is -ve (as \( v \to \infty \) \( \xi \to \xi_\infty \)). Therefore, \( v^2 \) increases with decreasing \( r \). At small \( r \), the numerator is -ve. 

If we want a solution such that \( v \) increases monotonically with decreasing \( r \), then we must have a +ve denominator i.e. \( v > \xi_\infty \) i.e. the flow is supersonic at small \( r \). Since the denominator is 0 at the sonic point \( (v = \xi_\infty) \), the numerator too must vanish here to keep the solution regular i.e.

\[ \text{at the sonic point} \quad \frac{\xi_\infty^2}{2} = \frac{6M}{2r} \quad (6) \]

Substituting this back into (2a) we find -

\[ \frac{\xi_\infty^2}{2} + \frac{\xi^2}{12} - 2\xi^2 = \frac{\xi_\infty^2}{12} \quad \text{or} \quad \frac{5 - 3\Gamma}{2} \xi^2 = \xi_\infty^2 \quad (7) \]

The accretion rate \( \dot{M} = \frac{4\pi r^2 \rho_\infty v_\infty}{\xi_\infty^2} = \frac{\pi (GM)^2 \rho_\infty (\xi_\infty^2)}{(\xi_\infty^2)^2} \quad (8) \]

\[ \dot{M} = \frac{\pi (GM)^2 \rho_\infty}{c_\infty^3} \left( \frac{5 - 3\Gamma}{2} \right) \quad (9) \]

\( \text{(subscript } \xi \text{ is for sonic point)} \)

**Note:** We can understand this result in a simple way - the rate above is what we obtain at the radius where the "free fall" starts i.e. \( \frac{GM}{r} \ll c_\infty^2 \Rightarrow \dot{M} \approx \rho_\infty r^2 c_\infty \approx \rho_0 (GM)^2 / c_\infty^3 \).

For \( M \approx 1M_\odot \), \( \rho_\infty \approx 1g/cm^3 \) & \( c_\infty \approx 10^8 cm/s \), \( \dot{M} \approx 10^{29} g/s \) which results in a luminosity of \( \approx 10^{39} \) erg/s if accreted onto a NS.
Note 1: The transonic accretion rate given by eq. (8) is the maximum possible accretion in a steady state. For smaller accretion rate, the speed is subsonic everywhere & vanishes in r=0. (See fig. 8a)

2: Solar/stellar winds is also described by the same set of equations, i.e., 1 & 2.

3: The accretion rate when the star moves with ISM with speed $V$ is $M \approx \frac{P_0 (GM)^2}{(V^2 + c_s^2)^3/2}$.

4: Behavior at small $r$: For $V < V_c$ we can neglect the RHS of (2a) since $c_s \gg c_{so}$ & balancing the r.h.s. negative term on LHS, we find $\frac{V^2 + c_s^2}{r} < \frac{G M}{r}$ (cs < V for supersonic flow) & the main conservation $\Rightarrow P \propto r^{-2} V^2$ at $r \rightarrow 0$.

5: There are no steady state transonic solutions for $T > 3/2$.
Roche Analysis

Potential in a rotating frame

The equation of motion in a rotating frame

\[
\frac{d\vec{V}}{dt} = \frac{d\vec{V}}{dt} + \vec{V} \times \vec{\omega} = -\nabla \Phi - 2 \vec{\Omega} \times \vec{V} - \frac{\vec{v} \times (\vec{v} \times \vec{r})}{r} \tag{coriolis, centrifugal force}
\]

For two point masses -

\[
\Phi = -\frac{GM_1}{r_{1}} - \frac{GM_2}{r_{2}} \quad \begin{cases} r_1, \quad r_2 \text{ are measured from the CM.} \\
\end{cases}
\]

\[
\vec{r} \times (\vec{v} \times \vec{r}) = (\vec{r} \cdot \vec{v}) \vec{r} - \vec{r} \times \vec{v} = -\vec{r} (\frac{1}{2} \vec{r} \cdot \vec{v}^2)
\]

\[
\nabla \times \vec{r} \times (\vec{v} \times \vec{r}) = \nabla \times (\vec{r} \cdot \vec{v}) \vec{r} = \nabla \vec{r} (\vec{r} \cdot \vec{v}) - \vec{r} \times (\nabla \times \vec{r} \vec{v}) = -\vec{r} \nabla (\frac{1}{2} \vec{r} \cdot \vec{v}^2)
\]

\[
\Phi_R = \Phi - \frac{1}{2} \vec{r} \cdot \vec{v}^2
\]

\[
\frac{d\vec{V}}{dt} = -\nabla \Phi_R - \frac{1}{2} \nabla \Phi_R 
\]

\[
\Rightarrow \frac{d^2}{dt^2} + \Phi_R - \frac{\vec{v}^2}{2} = E_J
\]

is conserved (E_J: Jacobi integral)

\[
L_1, L_2, \text{ and } L_3 \text{ are saddle points, and are unstable.}
L_4 \text{ and } L_5 \text{ are maximum but stable due to coriolis force.}
\]

It can be shown that (see Binney & Tremaine)

\[
E_J = E - \vec{r} \cdot \vec{L}^2
\]

\[
\text{angular energy in inertial frame momenta.}
\]
Accretion disk

1. Order of magnitude estimates

Let the accretion rate be $\dot{M}$. In steady state, $\dot{M}$ will pass through radius $r + dr$ & $r$ in time $dt$. The amount of energy released in time $dt$ between these radii is 

$$\frac{dE}{dt} = \frac{dM}{2r}$$

or 

$$\frac{dE}{dt} = \frac{GM\dot{M}}{dr}$$

If this energy release, or thermal energy, is radiated away in time $dt$ in the area between $r$ & $r + dr$ as blackbody radiation then,

$$6\pi T^4 (2\pi r^2 r) dt = \dot{M} \frac{r^2}{r^2 - 3/4}$$

or 

$$T = \left( \frac{GM\dot{M}}{4\pi R^2} \right)^{1/4}$$

For $M = 1 M_\odot$, $\dot{M} = M_{Edd}$, $r = 100 km$, $T = 4 \times 10^6 K$.

Observed spectrum:

Let us say that we make observation at frequency $\nu$. The part of the disk that will contribute to the flux at $\nu$ is with a radius $r_2$ such that

$$kT(r_2) = h \nu$$

for $r > r_2$, the contribution falls off exponentially. On $r_2 \propto \nu^{-3/4}$.

The observed flux 

$$f_\nu = 2\pi \int dr \frac{r^2 kT(r)}{C^2} \propto \nu^{2/3}$$

or 

$$f_\nu \propto \nu^{2/3}$$

Dwarf novae have this spectrum during outburst

(but not during quiescent period - probably because the disk is optically thin during quiet period).

Structure of Thin Accretion Disk

1. Mass conservation

The rate of mass inflow should be r-independent, i.e.
\[ 2\pi R \Sigma V_r = \dot{M} = \text{constant} \]
\[ \Rightarrow \text{surface mass density} \equiv \int_0^\infty \sigma \, d\Sigma = \text{constant} \]

2. Angular momentum conservation:

a. Inward flux of angular momentum carried by \( M \)
\[ \mathcal{L}_{\text{in}} = \dot{M} (GMr)^{3/2} \]  
(2a)

b. Outward flux due to viscous stress

Let \( \mathcal{G} \) be the torque associated with viscous stress that the disk within \( r \) exerts on the exterior disk. Then the rate of angular momentum transferred from the inner to the outer disk, i.e., the angular momentum flux, is \( \mathcal{G} \).

\[ \mathcal{G} = \mathcal{L}_{\text{in}} \text{ the net outward transport of angular momentum, which should be } r \text{-independent i.e.} \]
\[ \mathcal{G} = \dot{M} (GMr)^{3/2} = \text{constant} \]
\[ \text{if the torque vanishes at the inner edge of disk, e.g., for } r = \text{BHT, then} \]
\[ \mathcal{G} = \dot{M} (r_0^2 - r_1^2) (GM)^{3/2} \]  
(2)

c. For local viscosity & shear proportional to the velocity gradient, the torque \( \mathcal{G} = -2\pi \int r \sigma \frac{d\Sigma}{dr} \) shear stress in force per unit area.

We neglect pressure gradient in the \( r \)-direction, be therefore \( \sigma \) has

Keplerian velocity profile i.e. \[ \frac{dr}{dr} = -\frac{3}{2} \frac{a}{r} = -\frac{3}{2} \frac{r}{a} \frac{GM}{r^2} \]
\[ G = \frac{3\pi}{2} \Sigma \sqrt{G M r} \approx M \sqrt{G M r} \]

\[ \text{using eq. (2) for } r \gg r_{\text{min}} \]

or \[ M = 3\pi \Sigma r \quad (3) \]

The viscosity \( \nu \) must be much greater than molecular viscosity to drive accretion at a rate needed to explain the luminosity observed for X-ray binary systems or AGB disks. Shakura & Sunyaev suggested that the viscosity is the disk's turbulent viscosity:

\[ \nu = \alpha \frac{c_s H}{4} \quad \text{vertical scale height} \quad (4) \]

\[ \alpha \quad \text{sound speed} \]

\[ \alpha \quad \text{parameter} \]

(Note: ordinary molecular viscosity is \( \nu = c_s \lambda \), where \( \lambda \) is mean free path length for particle.)

3. **Energy conservation**

a. Gravitational energy release by accreting gas between \( r \) & \( r + \delta r \)

\[ dE = -M \left( \frac{dG M}{d\ln r} \right) \delta r = -M \frac{dG M}{d\ln r} \delta r \]

\[ \frac{dE}{dr} = \frac{G M m}{2r^2} \quad (5a) \]

b. The work done by the viscous torque on the ring between \( r \) & \( r + \delta r \)

\[ \frac{dE}{dr} = -\frac{dG M}{dr} \delta r \]

The contribution to the energy release by viscous torque:

\[ \frac{dE}{dr} = -\frac{dG M}{dr} \quad (5b) \]

Adding the above two contributions:

\[ \frac{dE}{dr} = \frac{G M m}{2r^2} - \frac{dG M}{dr} \quad (5) \]
Making use of eq. (2) for $G$ into (5) -

\[ \frac{dE}{dr} = \frac{G M}{2r^2} \left[ \ln\left(\frac{r'}{r}\right) \right] \]

\[ = \frac{G \mu M}{2r^2} \left[ -\frac{1}{2} + \frac{3}{2} \frac{r_{\text{min}}}{r^{3/2}} \right] \]

\[ \frac{dE}{dr} = \frac{3G \mu M}{2r^2} \left[ 1 - \left(\frac{r_{\text{min}}}{r}\right)^{3/2} \right] \]  

(6)

Integrating this over $r$ from $r_{\text{min}}$ to $\infty$ - we find that the total energy release is $G \mu M$, as expected. For $r \gg r_{\text{min}}$

\[ \frac{dE}{dr} = \frac{3G \mu M}{2r^2} \]

Therefore, twice as much energy is released as a result of viscous torque as the decrease in the PE of infalling gas!

The energy release per unit area

\[ Q^+ = \frac{\frac{1}{4\pi r^2} \frac{dE}{dr}}{4\pi r^3} \]

\[ = \frac{3G \mu M}{4\pi r^3} \left[ 1 - \left(\frac{r_{\text{min}}}{r}\right)^{3/2} \right] \]

(7)

4. Radiation

For optically thick disk the energy loss rate/area -

\[ Q \sim \frac{8 \pi c T^4}{3} \]

(8)

$T$ is temperature in disk mid plane

where $\tau = \kappa \Sigma$ is the optical depth of the disk in vertical direction; \( \kappa \approx 0.4 \text{ cm}^2\text{ g}^{-1} \) when opacity is dominated by Thomson scattering & $\approx 7 \times 10^{-22} \text{ cm}^{-2} \text{ s}^{-1} \text{ g}^{-1}$ for Rosseland opacity which is a combination of bound-free & free-free opacity for solar abundance.
5. Vertical disk structure

\[ \frac{dP}{dz} = -\rho g = -\rho \frac{a^2}{c^2} \]

for an isothermal structure in the vertical direction.

The scale height \( H = c_s / a \) \( (8) \)

\[ p = p_0 e^{-\frac{H^2 z^2}{2c^2}} \] \( (9) \)

\[ p = \frac{2kT}{\mu m_p} \frac{1}{a} \]

\[ a = 7.5 \times 10^{-15} \text{ erg cm}^{-2} \text{ s}^{-1} \text{ is the radiation constant} \]

**Disk Structure in steady state**

We will assume that heating rate is equal to radiative energy loss rate at all \( r \) locally (this does not have to be so).

The equations for thin disk (optically thin), are:

\[ \dot{M} = 2\pi r \Sigma v_r \] \( (1) \)

\[ \dot{M} = 3\pi \Sigma v \] \( (3) \)

\[ v = c_s H = c_s^2 / a \] \( (4) \)

\[ Q^* = \frac{a^2}{2} + 3\Sigma \mu m_p \eta = \frac{8acT^4}{3\pi \Sigma} \] \( (7) \) \& \( (8) \)

where \( \eta = 1 - \frac{\dot{M}}{\dot{M}_*} \)

We have 4 equations with 4 unknowns viz: \( E, v_r, T \& V \) and can solve these easily to determine \( E, T \& v_r \) as a function of \( r \).
Let us specialize to the case where the pressure is dominated by the gas molecules. In this case \( p = \frac{9kT}{4\pi m_R} \)

\[ S^2 = \frac{kT}{m_R \mu}. \]

The disk equations become:

\[
\dot{M} = 2\pi R v_r \Sigma
\]

\[
\dot{M} = \frac{3\pi \Sigma x c_s^2}{\alpha} = \frac{3\pi k x \Sigma T}{m_R \mu \alpha}
\]

\[
\frac{3M_0^2}{4\pi} = \frac{8acT^4}{3\Sigma \epsilon} = \frac{8acT^4}{3 \times 0.4 \Sigma}
\]

Eliminate \( \Sigma \) from (12) & (13) -

\[
\epsilon = 1 - \left( \frac{v_{\text{min}}}{v_r} \right)^2
\]

\[
\frac{3M_0^2}{4\pi} = \frac{8acT^5}{3\pi k \alpha}
\]

or \( T = \left[ \frac{1.2 M_0^2 \epsilon^2}{32 \pi^2 \alpha c k \alpha} \right]^{1/5} \sim 10^4 \text{K} \) in \( \left( \frac{M}{M_0} \right)^{3/5} \alpha^{-3/5} \epsilon^{-1/5} \)

\[
(14a)
\]

Notes: \( T \propto R^{-3/5} \), not \( T \propto R^{-3/4} \) derived earlier by balancing energy generation & loss rates, and the reason for this is that \( T \) given by (14a) is the temperature in disk midplane where as \( T \propto R^{-3/4} \) refers to the temperature above the disk mid-plane at the photosphere.

Substitute (14a) into (12) we find:

\[
p = \frac{\Sigma}{H} = \frac{\Sigma_0}{c_s} \alpha R^{-3/5}
\]

& from (11) \( v_R \alpha R^{-1/5} \).