5. Vertical disk structure

$$\frac{dp}{dz} = -\rho g = -\rho \Omega^2 z$$

for an isothermal structure in the vertical direction

The scale height $H = c_s/\Omega$ (8)

& $p = p_0 e^{-\frac{\Omega^2 z^2}{2c_s^2}}$ (9)

$$p = \frac{1}{3} a T^4 + \frac{\rho k T}{\mu m_p} \qquad (10)$$

μ — mean molecular weight

$a = 7.5\times10^{-15}$ erg cm^{-3} K^{-1} is the radiation constant

Disk Structure in steady state

We will assume that heating rate is equal to radiative energy loss rate at all r locally (this does not have to be so).
The equations for thin disk, (optically thick), are:

$$\dot{M} = 2\pi r \Sigma V_r \qquad (1)$$

$$\dot{M} = 3\pi \Sigma \nu \qquad (3)$$

$$\nu = \alpha c_s H = \alpha c_s^2/\Omega \qquad (4)$$

$$Q^+ = Q^- \text{ or } \frac{3\dot{M}\Omega^2}{4\pi}\eta = \frac{8acT^4}{3\kappa\Sigma} \qquad (7) \ \& \ (8)$$

where $\eta = 1 - \sqrt{\frac{r_{min}}{r}}$

We have 4 equations with 4 unknowns viz. Σ, V_r, T & ν & can solve these easily to determine Σ, T & V_r as a function of r.

Let us specialize to the case where the pressure is dominated by the gas pressure & $\varkappa$ by Thomson scattering. In this case $p = \frac{\rho k T}{m_p \mu}$ & $c_s^2 = \frac{kT}{m_p \mu}$. The disk equations become:

(This solution typically applies to middle disk; in the inner disk radiation pressure dominates; in outer disk gas pressure dominates but $\varkappa$ is due to free-free absorption).

$$\dot M = 2\pi r V_r \Sigma \qquad (11)$$

$$\dot M = 3\pi \Sigma \alpha c_s^2/\Omega = \frac{3\pi k \alpha \Sigma T}{m_p \mu \Omega} \qquad (12)$$

$$\frac{3 \dot M \Omega^2 \eta}{4\pi} = \frac{8 a c T^4}{3 \varkappa \Sigma} = \frac{8 a c T^4}{3 \times 0.4 \Sigma} \qquad (13)$$

$$\eta \equiv 1 - (r_{min}/r)^{1/2}$$

Eliminating Σ from (12) & (13) —

$$\rightarrow \quad \frac{3 \dot M \Omega^2 \eta}{4\pi} = \frac{8 a c T^5 3\pi k \alpha}{1.2 \dot M m_p \mu \Omega}$$

$$\rightarrow \quad \text{or } T = \left[\frac{1.2 \dot M^2 \Omega^3 \eta m_p \mu}{32\pi^2 a c k \alpha}\right]^{1/5} \sim 2.7\times10^7 K \, \dot m^{2/5} \left(\frac{M}{M_\odot}\right)^{\frac{7}{10}} r_7^{-\frac{9}{10}} \alpha^{-1/5} \qquad (14a)$$

mass accretion rate in units of Eddington rate

$$r_7 \equiv \left(\frac{r}{10^7 cm}\right)$$

Note: $T \propto r^{-9/10}$, NOT $T \propto r^{-3/4}$ derived earlier by balancing energy generation & loss rates, and the reason for this is that T given by (14a) is the temperature in disk midplane whereas $T \propto r^{-3/4}$ refers to the temperature above the disk mid-plane at the "photosphere".

$\rightarrow$ Substituting (14a) into (12) we find $\rho = \frac{\Sigma}{H} = \frac{\Sigma \Omega}{c_s} \propto \frac{\Omega^2}{T^{3/2}} \propto r^{-\frac{33}{20}}$

$\Sigma \propto r^{-3/5}$ & from (11) $V_r \propto r^{-2/5}$.

Thin disk

1. When is the thin disk approximation valid?

The equation for hydrostatic equilibrium in the vertical direction is:
$$\frac{dp}{dz} = -\rho g_z = -\rho\left(\frac{GMz}{r^3}\right) = -\rho z \Omega^2$$

For isothermal stratification in the vertical direction the equation can be easily solved to yield: $p = p_c e^{-\frac{z^2\Omega^2}{2c_s^2}}$
where $c_s^2 = p/\rho$.

The scale height $H = \frac{c_s}{\Omega} = \frac{r c_s}{r\Omega} = \frac{r c_s}{V_K}$ ← Keplerian rotation speed

$$\therefore \boxed{\frac{H}{r} = \frac{c_s}{V_K}}$$

The disk's thin, i.e. $H \ll r$, when $c_s \ll V_K$. Or in other words the thermal energy/particle is $\ll$ KE of rotation/particle. This means that for thin disk approximation to hold, most of the gravitational energy release must be radiated away.

2. Radiative efficiency: We want to find out if accretion energy release is radiated locally or not.

Calculate the disk structure assuming that energy release is radiated locally. If photons can be absorbed on their way out of the disk in the vertical direction then the disk can radiate locally as black-body & the disk structure we calculated assuming efficient radiation locally is valid.

The disk optical depth to Thomson scattering $\tau_T = \kappa_T \rho H$

where $\kappa_T \approx 0.4\ cm^2 g^{-1}$ is Thomson opacity.

Let the absorption opacity be $\varkappa_R$. (The Rosseland mean opacity is a reasonable approximation for $\varkappa_R$; $\therefore$ $\varkappa_R \sim 7\times10^{22}\rho T^{-3.5}\,cm^2g^{-1}$).

If $\tau_T \gg 1$, then the optical depth to photon absorption is —

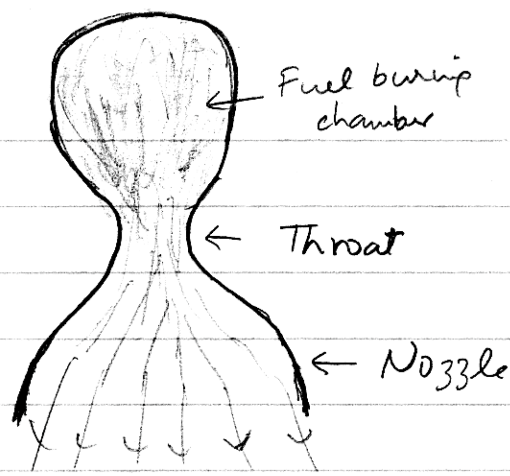
$$\tau_{abs} = \rho\varkappa_R\,\tau_T H = \varkappa_R\varkappa_T(\rho H)^2 = \varkappa_R\varkappa_T\Sigma^2 \;;\; \Sigma \equiv \rho H$$

The requirement for efficient radiation is $\tau_{abs} > 1$

or $\boxed{(\varkappa_R\varkappa_T)^{1/2}\Sigma > 1}$

(The effective opacity is the geometrical mean of scattering & absorption opacity.)

The above condition for efficient radiation breaks down in the inner part of the disk, a few tens of Schwarzschild radii or more precisely for $[r/(6M/c^2)] < 10\,\alpha^{1/3} M_8^{-2/3}\dot{M}_{25}^{2/3}$ (where $M_8 = M/10^8M_\odot$ & $\dot{M}_{25}$ is accretion rate in unit of $10^{25}\,g\,s^{-1}$), & the disk becomes geometrically thick.



Rocket Propulsion

Equations: mass conservation: $\rho A V = \dot{M}$ (1)

Momentum conservation: $V \dfrac{dV}{dX} + \dfrac{1}{\rho}\dfrac{dp}{dX} = 0$

for a polytropic gas $V\dfrac{dV}{dX} + c_s^2 \dfrac{d\ln\rho}{dX} = 0$ (2)

from (1): $\dfrac{d\ln\rho}{dX} + \dfrac{d\ln V}{dX} + \dfrac{d\ln A}{dX} = 0$ (3)

eliminate ρ from (2) & (3)

$$\frac{dV}{dX}\left(V - \frac{c_s^2}{V}\right) = + c_s^2 \frac{d\ln A}{dX}$$

or $\boxed{\dfrac{dV}{dX} = + \dfrac{V c_s^2}{V^2 - c_s^2}\dfrac{d\ln A}{dX}}$ (4)

For $V < c_s$: $\frac{dA}{dx} < 0 \Rightarrow \frac{dv}{dx} > 0$

$V > c_s$: $\frac{dA}{dx} > 0 \Rightarrow \frac{dv}{dx} > 0$

- So you must have $V = c_s$ at the throat where $\frac{dA}{dx} = 0$ in order to get the exhaust to supersonic speed & maximize the thrust.
- If A were to decrease monotonically with x i.e. no nozzle then the maximum exhaust speed is c_s.
- Thrust $= \dot{M} V_{exhaust}$ so you want $V_{exhaust}$ as large as possible.

The highest possible speed $V^{(max)}_{exhaust} = \sqrt{\frac{2}{\Gamma - 1}}\, c_{s,0}$ where $\begin{cases} c_{s,0}: \text{sound speed in the combustion chamber where } V=0 \\ \Gamma \text{ is the ratio of specific heats.} \end{cases}$

Efficiency of accretion onto a Schwarzschild blackhole

$$-d\tau^2 = -dt^2\left(1-\frac{2M}{r}\right) + \frac{dr^2}{1-\frac{2M}{r}} + r^2 d\theta^2 \qquad (1)$$

Particle orbit:

t-component of the Geodesic equation (or Euler-Lagrange equation) is:

$$-\frac{d}{d\tau}\left[\dot{t}(1-2M/r)\right] = 0 \Rightarrow \dot{t}(1-2M/r) = E \qquad (2)$$

(E is the energy measured by an observer at infinity) for unit rest mass particle

Energy measured by a local rest frame observer $= \dot{t}(1-2M/r)^{1/2} \equiv E_{local}$

$$\therefore \quad E = (1-2M/r)^{1/2} E_{local} \qquad (3)$$

θ-component of equation –

$$\frac{d}{d\tau}(r^2\dot{\theta}) = 0 \Rightarrow r^2\dot{\theta} = \mathcal{L} \qquad (4)$$

angular momentum per unit mass

Substituting (2) & (4) into (1) we find –

$$\dot{r}^2 = E^2 - \left(1+\frac{\mathcal{L}^2}{r^2}\right)(1-2M/r) \qquad (5)$$

define effective potential to be $V_{eff} = \left(1+\frac{\mathcal{L}^2}{r^2}\right)(1-2M/r) \qquad (6)$

The extremum of the potential is given by $\frac{dV_{eff}}{dr} = 0$

$$\text{or} \quad \frac{2M}{r^2} - \frac{2\mathcal{L}^2}{r^3} + \frac{6M\mathcal{L}^2}{r^4} = 0 \quad \text{or} \quad Mr^2 - r\mathcal{L}^2 + 3M\mathcal{L}^2 = 0 \qquad (7a)$$

$$\text{or} \quad r = \frac{\mathcal{L}^2 \pm \sqrt{\mathcal{L}^4 - 12M^2\mathcal{L}^2}}{2M} \qquad (7)$$

There are no minimum for V_{eff} for $\mathcal{L} < \sqrt{12}\, M \qquad (8)$

For stable circular orbit $\frac{d^2V_{eff}}{dr^2} > 0 \Rightarrow r > 6M \qquad (9)$

For circular orbits $\frac{dV_{eff}}{dr} = 0$ & $\dot{r} = 0$

from (7a) $\mathcal{L}^2 = \frac{Mr^2}{r-3M} \qquad (10)$

$$\& \ (5) \Rightarrow \quad E^2 = \left(1+\frac{\mathcal{L}^2}{r^2}\right)\left(1-\frac{2M}{r}\right) = \frac{(r-2M)^2}{r(r-3M)} = \frac{(1-2M/r)^2}{(1-3M/r)} \qquad (11)$$

For the last stable circular orbit $r = 6M$

$\therefore \quad E^2 = \frac{8}{9}$

or the binding energy of the last stable circular orbit $= 1 - \frac{\sqrt{8}}{3} = 5.7\%$

- The angular speed for a test particle on a circular orbit

$$\Omega = \frac{d\theta}{dt} = \frac{\dot{\theta}}{\dot{t}} = \frac{\mathcal{L}}{r^2} \frac{(1-2M/r)}{E}$$

using eqs. (10) & (11) for $\mathcal{L}$ & E for circular orbits —

$$\Omega = \frac{M^{1/2} r}{r^2 \sqrt{r-3M}} \frac{(1-2M/r)\, r^{1/2}\sqrt{r-3M}}{(r-2M)} = \sqrt{\frac{M}{r^3}}$$

Same as the Newtonian case!

Kerr Blackhole

The metric in Boyer-Lindquist coordinates is given by –

$$-d\tau^2 = -\left(1-\frac{2Mr}{\Sigma}\right)dt^2 - \frac{4aMr\sin^2\theta}{\Sigma}dt\,d\varphi + \frac{\Sigma}{\Delta}dr^2 + \Sigma\,d\theta^2 + \left[r^2+a^2+\frac{2Mra^2}{\Sigma}\sin^2\theta\right]\sin^2\theta\,d\varphi^2 \qquad (1)$$

where $a \equiv \frac{J}{M}$, $\Delta \equiv r^2 - 2Mr + a^2$, $\Sigma = r^2 + a^2\cos^2\theta$ (1a)

J is the angular momentum & M the mass of the BH.

$|a| \leq M$.

1. Let us calculate the allowed angular speed of a test particle (as seen from infinity) at a fixed r & θ.

The angular speed $\Omega = \frac{d\varphi}{dt} = \dot{\varphi}/\dot{t}$

from (1) $\quad -1 = \dot{t}^2\left[g_{tt} + 2g_{t\varphi}\Omega + g_{\varphi\varphi}\Omega^2\right]$

The quantity in the square bracket must be –ve, which means that Ω must lie in a certain range $[\Omega_-, \Omega_+]$

$$\Omega_{\mp} = \frac{-g_{t\varphi} \mp \sqrt{g_{t\varphi}^2 - g_{tt}g_{\varphi\varphi}}}{g_{\varphi\varphi}} \qquad (2)$$

(In flat-space time $-\frac{c}{r\sin\theta} < \Omega < \frac{c}{r\sin\theta}$)

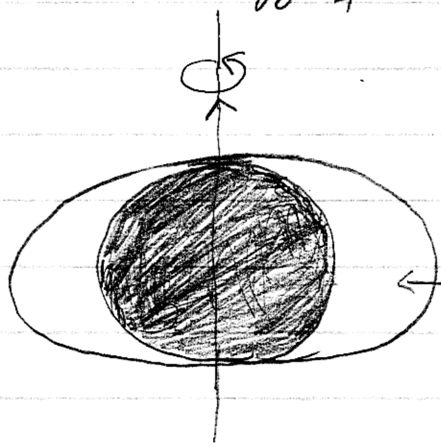
$\Omega_- = 0$ when $g_{tt} = 0$ or $r^2 + a^2\cos^2\theta - 2Mr = 0$

or $r_s = M + \sqrt{M^2 - a^2\cos^2\theta}$ (3)

For $r < r_s$ you have no static observer i.e. $\Omega = 0$ is not allowed & and everything is dragged around by the rotating BH.

2. The event ~~horizon occurs~~ where the function $\Delta = 0$ or $r_H = M + \sqrt{M^2 - a^2}$ (4)
$(r_H < r_s)$

The region of space between r_H & r_S is called the ergosphere. Energy can be extracted in this region (by tapping the rotational energy of the BH).



• Circular orbits in the equatorial plane

The metric in the equatorial plane is –

$$-d\tau^2 = -\left(1-\frac{2M}{r}\right)dt^2 - \frac{4aM}{r}dt\,d\varphi + \frac{r^2}{\Delta}dr^2 + \left(r^2+a^2+\frac{2Ma^2}{r}\right)d\varphi^2 \qquad (5)$$

t & φ components of the E-L equation –

$$\left(1-\frac{2M}{r}\right)\dot{t} + \frac{2aM}{r}\dot{\varphi} = E \qquad (6)$$

$$\frac{2aM}{r}\dot{t} - \left(r^2+a^2+\frac{2Ma^2}{r}\right)\dot{\varphi} = \mathcal{L} \qquad (7)$$

we can solve these equations for $\dot{t}$ & $\dot{\varphi}$ & substituting these back into eq. (5) we find $\dot{r}^2$ & therefore the effective potential

Some results from special relativity

- Transformation of 4-vector (A_0, A_x, A_y, A_z) — prime frame moving along x with velocity V & LF Γ. $(c=1)$

$$A_0' = \Gamma(A_0 + VA_x) \; ; \quad A_x' = \Gamma(A_x + VA_0), \quad A_y' = A_y \;\; \& \;\; A_z' = A_z$$

$$\Gamma \equiv (1-V^2)^{-1/2}$$

Doppler shift

Consider a photon with 4-momentum $(\underbrace{\nu}_{\text{Energy}}, \underbrace{\nu\cos\theta, \nu\sin\theta, 0}_{\text{3-momentum}}) = p^\mu$

$p^\mu p_\mu = 0$ (zero rest mass).

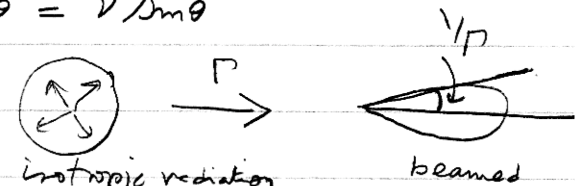
In the prime frame $p'^\mu = (\nu', \nu'\cos\theta', \nu'\sin\theta', 0)$

$$\nu' = \Gamma(\nu + V\nu\cos\theta) = \nu\Gamma(1 + V\cos\theta) = \nu\Gamma(1 + \vec{V}\cdot\hat{k})$$

Transverse doppler shift $\quad \nu' = \nu\Gamma \quad$ when $\vec{V}\cdot\hat{k} = 0$

$$\nu'\cos\theta' = \Gamma\nu(\cos\theta + V) \; ; \quad \nu'\sin\theta' = \nu\sin\theta$$

$$\therefore \quad \cos\theta' = \frac{\cos\theta + V}{1 + V\cos\theta}$$



$$\frac{d\Omega'}{d\Omega} = \frac{d\cos\theta'}{d\cos\theta} = \frac{1}{1+V\cos\theta} - \frac{(\cos\theta + V)V}{(1+V\cos\theta)^2} = \frac{1-V^2}{(1+V\cos\theta)^2}$$

or $$\frac{d\Omega'}{d\Omega} = \frac{1}{\Gamma^2(1+V\cos\theta)^2} \equiv \frac{1}{\mathcal{D}^2} = \left(\frac{\nu}{\nu'}\right)^2$$

- $d^3x\, d^3k$ is **Lorentz invariant** (as is d^4x of course).

$d^3x' = \frac{1}{\Gamma} d^3x$; the volume element d^3k is at constant frequency ($\delta\nu = 0$), which is ok because you can have photons of same ν & different $\vec{k}$, transform as $d^3k' = \Gamma d^3k$ — $dk_x' = \Gamma(dk_x + V d\nu) = \Gamma dk_x$; $dk_y' = dk_y$ & $dk_z' = dk_z$

$$\therefore \quad \boxed{d^3x'\, d^3k' = d^3x\, d^3k}$$

- Specific intensity

$$I \equiv \frac{dE}{dt\,dA\,d\nu\,d\Omega}$$ (energy received in photons per unit time per unit area, per unit ν & solid angle).

$$E = n h\nu, \quad dt\,dA = d^3x, \quad \nu^2 d\nu\, d\Omega = d^3k$$

$$I = h\nu^3 \frac{dn}{d^3x\,d^3k} \Rightarrow \boxed{\frac{I}{\nu^3} \text{ is Lorentz Invariant}}$$

$$\frac{I'}{\nu'^3} = \frac{I}{\nu^3} \quad \text{or} \quad I' = I(\nu'/\nu)^3 = I\mathcal{D}^3$$

Relativistic Jets (moving toward us) appear much brighter than they are in comoving frame & jets moving in opposite direction are faint.

- <u>Relativistic transformation of power</u>

Let us consider a radiating object that has no dipole asymmetry in its rest frame. In time δt it radiates δE energy & no net momentum,

Power $P = \frac{dE}{dt}$

in another frame $\delta t' = \Gamma \delta t$

& $\delta E' = \Gamma(\delta E + v\,\delta p) = \Gamma \delta E$

loss of momentum $= 0$

$$\therefore \quad \boxed{P' = \frac{dE'}{dt'} = \frac{dE}{dt} = P}$$

<u>Application</u> — Synchrotron radiation (radiation by a relativistic electron in a magnetic field).

Larmor's formula —

$$P = \frac{2}{3}\frac{q^2}{c^3}a^2$$ a is acceleration.

In the instantaneous rest frame of the particle, there is an electric field $\vec{E}' = \Gamma \frac{\vec{V}}{c}\times\vec{B} = \Gamma \frac{V_\perp B}{c}$

The electron acceleration in this frame $a' = \frac{qE'}{m_e} = \frac{q\Gamma V_\perp B}{m_e}$

or $P = \frac{2}{3}\frac{q^2}{c^3}\frac{q^2\Gamma^2 V_\perp^2 B^2}{m_e^2} = \frac{2}{3}\frac{q^4}{m_e^2 c^3}\Gamma^2 V_\perp^2 B^2$

$\sigma_T = \frac{8\pi}{3}\frac{q^4}{m_e^2 c^4}$; σ_T : Thomson cross-section

averaging over electron pitch angles (assuming random orientation).

$$\langle V_\perp^2 \rangle = \frac{V^2}{4\pi}\int d\Omega \sin^2\alpha = \frac{2}{3}V^2$$

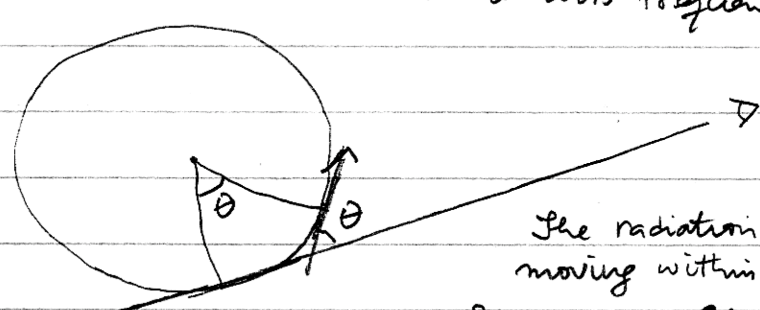
$\therefore \langle P \rangle = \frac{4}{3}\sigma_T \beta^2\Gamma^2 c \frac{B^2}{8\pi}$; $\beta \equiv \frac{v}{c}$

Synchrotron frequency

Equation of motion in a uniform magnetic field -

$\frac{d\Gamma m_e \vec{V}}{dt} = \frac{q}{c}\vec{V}\times\vec{B} \Rightarrow \Gamma\vec{V}\cdot\frac{d\Gamma\vec{V}}{dt} = 0$ or $\Gamma^2 V^2$ = constant or Γ is constant.

$\therefore \frac{d\vec{V}}{dt} = \frac{q\vec{V}\times\vec{B}}{m_e c\Gamma}$ this equation describes a circular (helical) orbit with frequency $\omega_B = \frac{qB}{m_e c\Gamma}$.



The radiation will be detected when e^- is moving within $1/\Gamma$ of the line of sight or

$$\delta t = \frac{2\pi}{\omega_B}\frac{2/\Gamma}{2\pi} = \frac{2}{\Gamma\omega_B}$$

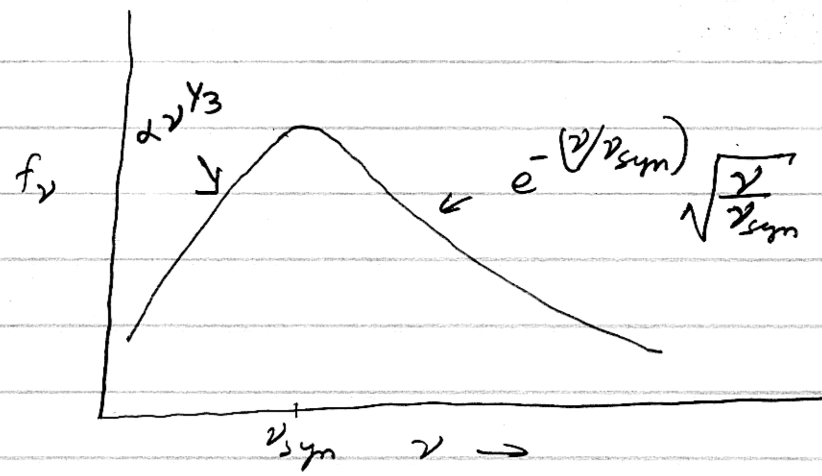
The time in the observer frame is smaller $\delta t_{obs} = \delta t(1-v) = \frac{\delta t}{2\Gamma^2}$

$\therefore \quad \delta t_{obs} = \dfrac{1}{\Gamma^3 \omega_B}$

The peak of the synchrotron spectrum is at $\omega_{syn} \approx \dfrac{1}{\delta t_{obs}}$

or $\quad \nu_{syn} \approx \dfrac{\Gamma^3 \omega_B}{2\pi} = \dfrac{\Gamma^2 q B}{2\pi m_e c}$

<u>Spectrum for a single electron</u>



- <u>Electron lifetime due to synchrotron radiation</u>

$$\frac{dE}{dt} = -\frac{4}{3}\sigma_T \frac{B^2}{8\pi}\gamma_e^2 c$$

$$E = m_e c^2 \gamma_e$$

$$\frac{d\gamma_e}{dt} = -\frac{\sigma_T B^2 \gamma_e^2}{6\pi m_e c} \quad \text{or} \quad \boxed{\gamma_e(t)^{-1} - \gamma_e^{-1}(t=0) = \frac{\sigma_T B^2 t}{6\pi m_e c}}$$

$$\text{or} \quad \frac{\gamma_e(t)}{\gamma_e(0)} \approx \frac{t_c}{t} \quad \text{where} \quad t_c \equiv \frac{6\pi m_e c}{\sigma_T B^2 \gamma_e}$$

Synchrotron radiation from a distribution of electrons.

Consider a powerlaw distribution of $\bar{e}$s —

$$\frac{dn_e}{d\gamma_e} = K\,\gamma_e^{-p} \qquad \text{for} \quad \gamma_{min} < \gamma_e < \gamma_{max} \qquad (1)$$

- The radiation at frequency ν comes from electrons with $\gamma_e > \gamma_\nu$ where γ_ν is given by —

$$\nu = \frac{qB}{2\pi m_e c}\gamma_\nu^2$$

- The flux contributed by an e^- at ν

$$\epsilon_\nu = \frac{\sigma_T B c}{6\pi}\,\frac{2\pi m_e c}{q}\left(\frac{\nu\, 2\pi m_e c}{qB\gamma_e^2}\right)^{1/3} \qquad (2a)$$

$$\epsilon_\nu = \frac{\sigma_T m_e B c^2}{3q}\left(\frac{2\pi m_e c\,\nu}{qB\gamma_e^2}\right)^{1/3} \qquad \gamma_e \geq \gamma_\nu \qquad (2b)$$

Therefore, the energy radiated per unit volume per unit time at ν is —

$$j_\nu = \int_{\gamma_\nu}^{\gamma_{max}} d\gamma_e\,\frac{dn_e}{d\gamma_e}\,\epsilon_\nu = K\,\frac{\sigma_T m_e c^2 B^{2/3}}{3q}\left(\frac{2\pi m_e c\,\nu}{q}\right)^{1/3}\int d\gamma_e\,\gamma_e^{-p-\frac{2}{3}}$$

$$\approx K\,\frac{\sigma_T m_e c^2 B^{2/3}}{3q(p-1/3)}\left(\frac{2\pi m_e c\,\nu}{q}\right)^{1/3}\gamma_\nu^{-p+1/3}$$

or $$j_\nu \approx K\,\frac{\sigma_T m_e c^2 B^{2/3}}{3q(p-1/3)}\left(\frac{2\pi m_e c\,\nu}{q}\right)^{1/3}\left(\frac{qB}{2\pi m_e c\,\nu}\right)^{\frac{p}{2}-\frac{1}{6}}$$

or $$\boxed{j_\nu \approx K\,\frac{\sigma_T m_e c^2}{3q}\left(\frac{q}{2\pi m_e c}\right)^{\frac{p-1}{2}} B^{\frac{p+1}{2}}\,\nu^{-\frac{p-1}{2}}} \qquad (3)$$

or $$j_\nu \approx K\,\frac{\sigma_T c}{6\pi}\left(\frac{2\pi m_e c}{q}\right)^{\frac{3-p}{2}} B^{\frac{p+1}{2}}\,\nu^{-\frac{p-1}{2}}$$

Minimum energy argument for Synchrotron sources

Let us consider a synchrotron source that is resolved, i.e. we know its angular size, and its known distance — so we know the volume of the source. Let us assume that we know the spectrum, the flux at a frequency & the frequency range over which the source spectrum is a power law.

Using these observables we can determine the minimum energy requirement for the source — the sum of magnetic & electron energies.

Energy in relativistic electrons:

$$E_e = V \int_{\gamma_{min}}^{\gamma_{max}} \frac{dn_e}{d\gamma_e} \gamma_e m_e c^2 \, d\gamma_e = \frac{V K m_e c^2}{p-2} \left(\gamma_{min}^{2-p} - \gamma_{max}^{2-p} \right) \quad (4)$$

Energy in magnetic field —

$$E_B = V \frac{B^2}{8\pi} \quad (5)$$

where V is the source volume.

The total energy $E = E_B + E_e = V \left[\frac{B^2}{8\pi} + \frac{K m_e c^2}{p-2} \left(\gamma_{min}^{2-p} - \gamma_{max}^{2-p} \right) \right]$ (6)

(Note: proton energy should also be included in the calculation of total energy. However, p^+ energy is very difficult to constrain & it is normally assumed to be $\propto E_e$.)

We have 4 unknowns in the expression for E (eq. 6) viz. K, B, γ_{min} & γ_{max}. We will try to eliminate 3 of these in terms of B.

K can be eliminated using eq. (3); note that $V j_\nu / 4\pi d^2 = f_\nu$ the flux we observe at frequency ν; so everything other than B is known quantity in eq. (3).

γ_{min} & γ_{max} are determined from the minimum & maximum frequencies (ν_{min} & ν_{max}) of the observed spectrum (in practice we typically only determine the upper & lower limits for ν_{min} & ν_{max} & therefore γ_{min} & γ_{max}).

$$\nu_{min} = \frac{q B \gamma_{min}^2}{2\pi m_e c} \quad \text{or} \quad \left.\begin{aligned} \gamma_{min} &= \left[\frac{2\pi m_e c\, \nu_{min}}{q B}\right]^{1/2} \\ \gamma_{max} &= \left[\frac{2\pi m_e c\, \nu_{max}}{q B}\right]^{1/2} \end{aligned}\right\} \qquad (7)$$

Substituting (3) & (7) into (6) we find –

$$E = V\left[\frac{B^2}{8\pi} + \eta f_\nu \nu^{\frac{p-1}{2}} B^{-3/2}\left(\nu_{min}^{-\frac{p-2}{2}} - \nu_{max}^{-\frac{p-2}{2}}\right)\right] \qquad (8)$$

where $$\eta \equiv \frac{12\pi d^2 q}{V(p-2)}\left(\frac{2\pi m_e c}{q}\right)^{\frac{1}{2}} \qquad (9)$$

Eq. (8) contains only one unknown (B) & to find the minimum energy we set $dE/dB = 0$ i.e. -

$$\frac{B}{4\pi} = \frac{3}{2}\eta f_\nu \nu^{\frac{p-1}{2}} B^{-5/2}\left(\nu_{min}^{-\frac{p-2}{2}} - \nu_{max}^{-\frac{p-2}{2}}\right)$$

or $$\frac{B^2}{8\pi} = \frac{3}{4} \underbrace{\eta f_\nu \nu^{\frac{p-1}{2}} B^{-3/2}\left(\nu_{min}^{-\frac{p-2}{2}} - \nu_{max}^{-\frac{p-2}{2}}\right)}_{= E_e/V} \qquad (10)$$

$\therefore$ The for the minimum energy solution $E_B = \frac{3}{4} E_e$ (a rough equipartition between magnetic & electron energies).

This technique is widely used to estimate energy in extended radio lobes ($E_{min} \approx 10^{59}$ erg!), supernova remnants, γ-ray bursts etc